Approximation of continuous real valued functions

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ABSTRACT

The aim of the article is to trace with some details the history of approximation of functions in C [a,b] beginning with weierstrars and the constructive theory initiated by Bernstein.

Key words: Supremum norm, metric space, Banach space, sequence of polynomials, normed linear space.

Introduction:

Weierstrars in 1985 showed that a continuous real function defined on a compact interval can be approximated uniformly to any desired extent by a polynomial. Bernstein later showed that knowing the desired nearness of the polynomial to a function one could construct it. If [a,b] is the compact interval and C [a,b] is the space of all continues functions on [a,b] and P the space of all polynomials with real coefficients, then P is a dense linear subspace of the Banach space C [a,b] with the norm

$$\| f \| = \sup_{a \le x \le b} | f(x) |, f \varepsilon c[a,b]$$

In other words, given an f ε C [a,b], there exists for every ε>0, a p ε P which is near to f by less than ε.

Weirstrars theorem:

Let f be in C [a,b]. Then given ε > 0, there exist a polynomial P such that

$$| f(x) - p(x)| < \varepsilon , (a \le x \le b) \text{ ------------ > (1)}$$

Remark:

C [a,b] is a complete metric space with metric ρ defined by

$$\rho (f, g) = \|f-g\| \quad (f, g \varepsilon C[a , b] \text{ -------> (2)}$$

Where ||f|| is called the supremum norm of *f* defined by

$$\| f \| = \max_{a \le x \le b} | f(x) | \text{ --------- > (3)}$$

(1) Is equivalent to the statement

There exists a sequence $\{p_n\}$ ε P such that p_n converges to f unifromely on [a,b].

(i.e) If p_n is chosen such that

$|f(x) - p_n(x)| < \frac{1}{n}$, $a \le x \le b$, then $p_n \to f$ uniformly on [a,b]

Conversely if ε>0 is given, we need to choose n with $\frac{1}{n} < \varepsilon$, a p_{n_0} with

$|f(x) - p_{n_0}(x)| < \frac{1}{n}$

Thus the theorem can be restated depending on the context.

Bernstein Proof of the theorem:

Take a=0 , b=1

Let [a,b] be any closed bounded interval.

Let f ε c [a , b]

Let g(x) = f[a + (b-a)x], 0≤x≤1

Now g(0) = f(a) , f(1) = f(b).

Clearly g ε c [0,1]

Hence there exists a polynomial Q such that |g(y) – Q(y)| < ε , 0≤y≤1

If $y = \frac{x-a}{b-a}$, then

$$g(y) = g\left(\frac{x-a}{b-a}\right)$$

$$= f\left(a + (b-a)\left(\frac{x-a}{b-a}\right)\right) = f(x)$$

Thus $\left| f(x) - Q\left(\frac{x-a}{b-a}\right) \right| < \in$, a ≤ x ≤ b -----------------→ (4)

If $p(x) = Q\left(\frac{x-a}{b-a}\right)$, then p is a polynomial as Q is a polynomial consider C [0,1].

For any fεC[0,1] we define a sequence of polynomials Bn,n=1,2,……….. as follows .

$$Bn(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ 0 \le x \le 1 \quad \text{------------}\to(5)$$

Where $\binom{n}{k}$ represents the number of k combinations out of n.

B_n is called the n^{th} Bernstein polynomial for f.

Given ε>0,we show that there exists n ε N such that

||f- Bn||<ε (n≤ N)

For any p ,q ε R, by Binomial theorem

$$\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = (p+q)^n, n \in N \text{ -----------}\rightarrow(6)$$

Differentiate with respect to p we get

$$\sum_{k=0}^{n}\binom{n}{k}kp^k q^{n-k} = n(p+q)^{n-1},$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{k}{n}p^{k-1} q^{n-k} = p(p+q)^{n-1}, \text{ ---------------}\rightarrow(7)$$

Differentiate again with respect to p,

$$\sum_{k=0}^{n}\frac{k^2}{n}\binom{n}{k}p^{k-1} q^{n-k} = p(n-1)(p+q)^{n-2} + (p+q)^{n-1}$$

$$(i.e)\sum_{k=0}^{n}\frac{k^2}{n^2}\binom{n}{k}p^{k} q^{n-k} = p^2\left(1-\frac{1}{n}\right)(p+q)^{n-2} + \frac{p}{n}(p+q)^{n-1} \text{ -----------}\rightarrow(8)$$

Sub p=x, q=1-x , 0<x<1

Then (6), (7)& (8) becomes

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = 1$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = x$$

$$\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)\binom{n}{k}x^k(1-x)^{n-k} = x^2\left(1-\frac{1}{n}\right)+\frac{x}{n} \text{ ---------------------}\rightarrow(9)$$

From (9) we have

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)\binom{n}{k}x^k(1-x)^{n-k} - 2x\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} + x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$$

$$= x^2 - \frac{x^2}{n} + \frac{x}{n} - 2x(x) + x^2$$

$$= \frac{x(1-x)}{n}$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \left(\frac{x(1-x)}{n}\right) \text{ -----------}\rightarrow(10)$$

Now f ε C[0,1] is uniformly continuous on the compact interval [0,1].

Hence given ε>0, there exists a δ>0

Such that |f(x)-f(y)|< ε/2, Whenever

|x-y|< δ, x, y ε[0,1]

Assuming ||f||≠0, we have N such that $\frac{1}{\sqrt[4]{N}} < \delta$ ----------→(11)

$$\frac{1}{\sqrt{N}} < \frac{\in}{4\|f\|} \quad \text{----→(12)}$$

For fixed $x \in [0,1]$ we have

f(x)- Bn(x) = $\sum_{k=0}^{n} | f(x) - f\left(\frac{k}{n}\right) | \binom{n}{k} x^k (1-x)^{n-k}$

$$= \sum{}' + \sum{}'' \text{ --------------→(13)}$$

Where $\sum{}'$ is the sum over there values of k such that

$$\left|\frac{k}{n} - x\right| < \frac{1}{\sqrt[4]{n}} \text{ ---------→(14)}$$

And $\sum{}''$ is the sum over the other values of k for which

$$\left|\frac{k}{n} - x\right| < \frac{1}{\sqrt[4]{n}}$$

$$(k-nx)^2 = n^2 \left|\frac{k}{n} - x\right|^2 \geq \sqrt{n^3}$$

Hence $|\sum{}''| = |\sum{}'' [f(x) - f\left(\frac{k}{n}\right)] \binom{n}{k} x^k (1-x)^{n-k}|$

$$\leq \sum{}'' [|f(x)| + |f\left(\frac{k}{n}\right)|] \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq 2\|f\| \sum{}'' \binom{n}{k} x^k (1-x)^{n-k} |$$

$$\leq 2\frac{\|f\|}{\sqrt{n^3}} \sum{}'' (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq 2\frac{\|f\|}{\sqrt{n^3}} \sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$\therefore |\sum{}''| \leq 2\frac{\|f\|}{\sqrt{n^3}} nx(1-x) \leq \frac{2\|f\|}{\sqrt{n}}$$

If $n \geq N$ it follows from (12) that

$\frac{1}{\sqrt{n}} < \frac{\in}{4\|f\|}$ and so

$|\sum{}'' | < \frac{\in}{2}$

Moreover if n≥ N and if k refer(14) then by (11), $|\frac{k}{n} - x| < \frac{1}{\sqrt[4]{N}}$ and so

$|f(x)-f(\frac{k}{n})| < \frac{\in}{2}$

$$|\sum{}' | = |\sum{}' [f(x) - f\left(\frac{k}{n}\right)]\binom{n}{k}x^k(1-x)^{n-k}|$$

$$< \frac{\in}{2}\sum{}' \binom{n}{k}x^k(1-x)^{n-k}$$

$$< \frac{\in}{2} \text{ by (9)}$$

Hence $|f(x)-Bn(x)| \le |\sum{}' | + |\sum{}'' |$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Conclusion:

The aim of the article is to express how a classical notation has given rise to rich abstractions.

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