COMPARISON OF TAYLOR AND PADÉ-2 APPROXIMATIONS IN OPTIMAL CONTROL PROBLEM FOR A DISTRIBUTED PARAMETER SYSTEM.

Mai Trung Thai[1]

[1]*Thai Nguyen University of Technology, Thai Nguyen city, Viet Nam*

ABSTRACT

The delayed control objects often meet in many different fields such as industry, transport, transportation, military... Normally, when designing the controller, if the object is the delayed first order inertia system which is approximated by two systems of the first order inertia, this often leads to the large error if the delayed time (τ) is significantly large compared to its time constant (T). This paper considers a delayed object $e^{-\tau s}$ *which has the ratio* T/τ *satisfies the condition* $2 \le T/\tau < 6$ [4,7]. *So, we presents a research comparing the accuracy of the solution when replacing a delayed object by Taylor approximation model and second-order Padé approximation model (Padé-2) in order to solve the optimal control problem for a distributed parameter system with delayed time. The system is also applied to a specific one-sided heat-transfer system in a heating furnace to control temperature for a flat-slab following the most accurate burning standards.*

Keyword: - *optimal control, distributed parameter system, delay, numerical method, Taylor approximation, Padé approximation.*

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1. INTRODUCTION

Theoretically, Taylor approximation and Padé approximation [1] have been studied for a long time and its mainly applycation is to find the solution of differential algebraic equations. Padé approximation can be offered the functional approximation having more advantages than Taylor expansion, especially with objects have large delayed time (τ) compare to its time constant (T) [4,7,8].

The paper has still continued to develop in some previous studies as in [4,5.6,7,8..], in which, we give two replacing methods for a delayed object by using Taylor approximation model and Padé-2 approximation model in order to solve the optimal control problem for a distributed parameter system with delayed time, typically for delayed objects with distributed parameter is heat transfer process.

Algorithms and simulation results have shown that depending on the relationship between (τ) and (T), which approximation form is best to use.

2. THE PROBLEM OF OPTIMAL CONTROL

2.1. The object model

As a typical distributed parameter system, the one-sided heat conduction system is considered. The process of one-sided heating of the objects which have flat-slab shape in a furnace is described by the parabolic-type partial differential equation, as follows in [2,5,6,7,8].

$$a\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t} \tag{1}$$

where $q(x,t)$, the temperature distribution in the object, is the output needing to be controlled, depending on the spatial coordinate x with $0 \le x \le L$ and the time t with $0 \le t \le t_f$, a is the temperature-conducting factor (m^2/s), L is the thickness of object (m), t_f is the allowed burning time (s)

The initial and boundary conditions are given in [2,5,6,7,8].

$$q(x,0) = q_0(x) = \text{const} \tag{2}$$

$$\lambda \frac{\partial q(x,t)}{\partial x}\bigg|_{x=0} = \alpha\left[q(0,t) - v(t)\right] \tag{3}$$

$$\frac{\partial q(x,t)}{\partial x}\bigg|_{x=L} = 0 \tag{4}$$

with α as the heat-transfer coefficient between the furnace space and the object (W/m^2.^{0}C), λ as the heat-conducting coefficient of material (W/m.^{0}C), and *v(t)* as the temperature of the furnace respectively (^{0}C).

The relationship between the provided voltage for the furnace *u(t)* and the temperature of the furnace *v(t)* is usually the first order inertia system with delayed-time as in [2,5,6,7,8].

$$T.\frac{dv(t)}{dt} + v(t) = k\,.u(t-\tau) \tag{5}$$

where *T* is the time constant; τ is the delayed time; *k* is the static transfer coefficient; *v(t)* is the temperature of the furnace and *u(t)* is the provided voltage for the furnace (controlled function of the system).

2.2 The objective function and the constrained conditions

In this case, the problem is set out as follows: we have to determine a control function *u(t)* with ($0 \le t \le t_f$) so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It means at the end of the heating process to ensure temperature uniformity throughout the whole material:

$$J\left[u(t)\right] = \int_0^L \left[q^*(x) - q(x,t_f)\right]^2 dx \to \min \tag{6}$$

The constrained condition of the control function:

$$U_1 \le u(\mathrm{t}) \le U_2 \tag{7}$$

with U_1, U_2 are the under and upper limit of the supplied voltage respectively (V).

3. THE SOLUTION OF PROBLEM

The process of finding the optimal solution includes 2 steps:

- S*tep 1*: Find the relationship between *q(x,t)* and the control signal *u(t)*. Namely, we have to solve the equation of heat transfer (relationship between *v(t)* and *q(x,t)*) with boundary condition type-3 combined with ordinary differential equation with delayed time (relationship between *u(t)* and *v(t)*)

- S*tep 2*: Find the optimal control signal $u^*(t)$ by substituting *q(x,t)* found in the first step into the function (6), after that finding optimal solution $u^*(t)$.

3.1. Find the relationship between q(x,t) and the control signal u(t)

To solve the partial differential equation (1) with the initial and the boundary conditions (2), (3), (4), we apply the Laplace transformation method with the time parameter *t*. On applying the transform with respect to *t*, the partial differential equation is reduced to an ordinary differential equation of variable *x*. The general solution of the ordinary differential equation is fitted to the boundary conditions, and the final solution is obtained by the application of the inverse transformation.

Transforming Laplace (1), we obtained:

$$a\frac{\partial^2 Q(x,s)}{\partial x^2} = sQ(x,s) \tag{8}$$

where: $Q(x,s) = \mathbf{L}\{q(x,t)\}$

After transforming the boundary conditions (3), (4), we have:

$$\lambda \frac{\partial Q(x,s)}{\partial x}\bigg|_{x=0} = \alpha\left[Q(0,s) - V(s)\right] \tag{9}$$

$$\frac{\partial Q(x,s)}{\partial x}\bigg|_{x=L} = 0 \tag{10}$$

To solve this problem, [2,6] replaced delayed object in Eq. (5) by the first order inertia system following Taylor approximation, [4] replaced delayed object in Eq. (5) by Padé-2 approximation. Transforming Laplace Eq. (5), we obtained:

- Following Taylor, Eq. (5) becomes:

$$(Ts+1)V(s) = k.U(s).e^{-\tau s} \cong k.\frac{U(s)}{1+\tau s} \tag{11}$$

- Following Padé-2, Eq. (5) becomes:

$$(Ts+1)V(s) = k.U(s).e^{-\tau s} \cong k.U(s).\frac{12-6\tau s+\tau^2 s^2}{12+6\tau s+\tau^2 s^2} \tag{12}$$

where

$$V(s) = \mathcal{L}\{v(t)\}\ ;\ \ U(s) = \mathcal{L}\{u(t)\} \tag{13}$$

The general solution of (1) is:

$$Q(x,s) = A(s).e^{\sqrt{\frac{s}{a}}.x} + B(s).e^{-\sqrt{\frac{s}{a}}.x} \tag{14}$$

where: *A(s); B(s)* are the parameters need to be find. After transforming, we have the functions:

- Function *Q(x,s)* (following Taylor approximation)

$$Q(x,s) = U(s)\frac{k.ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1)(\tau s+1)\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}sh\sqrt{\frac{s}{a}}.L + ch\sqrt{\frac{s}{a}}.L\right)} \tag{15}$$

Putting

$$G(x,s) = \frac{k.ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1)(\tau s+1)\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}sh\sqrt{\frac{s}{a}}.L + ch\sqrt{\frac{s}{a}}.L\right)} \tag{16}$$

We have: *Q(x,s) = G(x,s) .U(s)* (17)

- Function *Q(x,s)* (following Padé-2 approximation)

$$Q(x,s) = \frac{U(s).k.\left(12-6\tau s+\tau^2 s^2\right)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)} + e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)\left(12+6\tau s+\tau^2 s^2\right)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L} + e^{\sqrt{\frac{s}{a}}.L}\right] - \lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L} - e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \tag{18}$$

Putting

$$G(x,s) = \frac{k.\left(12-6\tau s+\tau^2 s^2\right)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)} + e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)\left(12+6\tau s+\tau^2 s^2\right)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L} + e^{\sqrt{\frac{s}{a}}.L}\right] - \lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L} - e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \tag{19}$$

We also have: *Q(x,s) = G(x,s) .U(s)* (20)

From (17) and (20), according to the convolution theorem, the inverse transformation of (17) and (20) is given by

$$q(x,t) = g(x,t) * u(t) \tag{21}$$

We can write

$$q(x,t)=\int_0^t g(x,\tau).u(t-\tau)d\tau \qquad (22)$$

or
$$q(x,t)=\int_0^t g(x,t-\tau).u(\tau)d\tau \qquad (23)$$

where
$$g(x,t)=\mathrm{L}^{-1}\left\{G(x,s)\right\} \qquad (24)$$

Therefore, if we know the function *g(x,t)*, we will be able to calculate the temperature distribution *q(x,t)* from control function *u(t)*. To find *q(x,t)* in (23), we need to find the function (24). Using the inverse Laplace transformation of function *G(x,s)* we have the following result:

- Function g*(x,t)* (following Taylor)

$$g(x,t)=\frac{k.k_0^2.\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(1-\tau k_0^2\right)\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0{}^2t}+$$

$$+\frac{k.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+$$

$$+\sum_{i=2}^{\infty}\frac{2\alpha.k.\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i{}^2\right)\left(1-\tau.\Psi_i{}^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i{}^2t} \qquad (25)$$

with $k_0=1/\sqrt{T};\ k_1=1/\sqrt{\tau};$

- Function g*(x,t)* (following Padé-2)

$$g(x,t)=\frac{k.k_0^2\left(12+6\tau k_0^2+\tau^2k_0^4\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(12-6\tau k_0^2+\tau^2k_0^4\right).\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0{}^2t}+$$

$$+\frac{k.\left(12+6\tau k_1^2+\tau^2k_1^4\right).\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{2.\left(1-Tk_1^2\right)\left(3\tau-\tau^2k_1^4\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+$$

$$+\frac{k.\left(12+6\tau k_2^2+\tau^2k_2^4\right).\cos\left(\frac{k_2}{\sqrt{a}}(L-x)\right)}{2.\left(1-Tk_2^2\right)\left(3\tau-\tau^2k_2^4\right)\left[\cos\left(\frac{k_2L}{\sqrt{a}}\right)-\frac{\lambda k_2}{\alpha\sqrt{a}}\sin\left(\frac{k_2L}{\sqrt{a}}\right)\right]}.e^{-k_2^2t}+$$

$$+\sum_{i=3}^{\infty}\frac{2\alpha.k\left(12+6\tau.\Psi_i{}^2+\tau^2\Psi_i{}^4\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i{}^2\right)\left(12-6\tau.\Psi_i{}^2-\tau^2\Psi_i{}^4\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i{}^2t} \qquad (26)$$

with $k_0=\frac{1}{\sqrt{T}};\ k_1{}^2=\frac{3-\sqrt{3j^2}}{\tau};\ k_2{}^2=\frac{3+\sqrt{3j^2}}{\tau};$

In Eq. (25) and Eq. (26):

- α is the heat-transfer factor ($W/m^2.^0C$).
- λ is the heat-conducting factor of object ($W/m.^0C$).
- L is the thickness of object (m).
- a is the temperature-conducting factor (m^2/s).
- τ is the delayed time of the furnace (s).
- k is the static transfer coefficients of the furnace.
- T is the time constant of the furnace (s).
- Ψ_i is calculated from the formula:

$$\Psi_i = \phi_i \sqrt{a} \,/\, L \tag{27}$$

- ϕ_i is the solution of the equation:

$$\phi_i .tg\phi_i = \alpha L / \lambda = B_i \tag{28}$$

- B_i is the coefficient BIO of the material.

Conclusions:

We have solved a system of parabolic-type partial differential equation with boundary conditions of type-3 (the relationship between *v(t)* and *q(x,t)*) combined with the ordinary differential equation with time delay (the relationship between *u(t)* and *v(t)*).

Thus, if we are not interested in the optimal problem, we can calculate the temperature field in the object when knowing the supplied voltage for the furnace (The problem knows the shell to find the cores), as follows:

The relationship between the supplied voltage for the furnace u(t) and the temperature field distribution in the object q(x,t):

$$q(x,t) = g(x,t) * u(t) = \int_0^t g(x,t-\tau).u(\tau)d\tau \tag{29}$$

3.2. Find the optimal control signal $u^*(t)$ by using numerical method

To find the $u^*(t)$, we have to minimize the objective function (6), it means:

$$\mathrm{J}\left[u(t)\right] = \int_0^L \left[q^*(x) - q(x,\mathrm{t}_f)\right]^2 dx \to \min \tag{30}$$

or

$$\mathrm{J}\left[u(t)\right] = \int_0^L \left[q^*(x) - \int_o^{t_f} g(x,\mathrm{t}_f - \tau)u(\tau)d\tau\right]^2 dx \to \min \tag{31}$$

with $q^*(x)$ is the desired temperature distribution; $q(x,t_f)$ is the real temperature distribution of the object at time $t = t_f$, with t_f is the allowed burning time (s).
As calculated in [2,5,6,7,8] the integral numerial method is used by applying Simson formula to the right-hand side of the objective function (31). The L, the thickness of the object, is divided into n equal lengths (n is an even number). Similarly, it is applied to the right-hand side of the equation (31). The period of time t_f is devided into m equal intervals that m is an even number, too.
Thus, the optimal control problem is here to find u^*_j in order to minimize the objective function:

$$\mathrm{J}[u^*] = L\sum_{i=0}^{n} \xi_i \left[q_i^* - \sum_{j=0}^{m} c_{\mathrm{ij}}.u_{\mathrm{j}}\right]^2 \tag{32}$$

The constrained conditions of the control function: $U_1 \le u_j \le U_2$ $(j = 0,1,2 \ldots m)$ (33)

The performance index (32) is a quadratic function of the variables u_j with constraints (33) are linear, the problem becomes a quadratic programming problem. This problem can be obtained by using numerical method after a finite number of iterations of computation.

4. SOME SIMULATION RESULTS

After building the algorithms and establishing the control programs, we have proceeded to run the simulation programs on a Diatomite sample in two cases Taylor approximation and Padé-2 approximation in order to test calculating programs.

4.1. Case 1: when delayed objects satisfy the condition: $T/\tau \geq 10$ in [5]

- The physical parameters of the object:

  $\alpha = 60\ (w/m^2.{}^0C);\ \lambda = 0.2\ (w/m.{}^0C);\ a = 3.6{*}e^{-7}(m^2/s);\ L = 0.04\ (m)$
- The parameters of the furnace: $T = 1200\ (s);\ \tau = 80\ (s;)\ k = 0.3$
- The desired temperature distribution: $q^* = 400^0C$
- The period of heating time: $t_f = 5400$ (s)
- Limit the temperature of furnace: $u(t) \leq 600^0$C
- Limit the temperature of flat-slab surface: $q(0,t) \leq 500\ {}^0$C
- Limit under voltage: $U_1=125$ (V)
- Limit upper voltage: $U_2=205$ (V)

After the simulation, we have results like in Figure 1 and Figure 2.

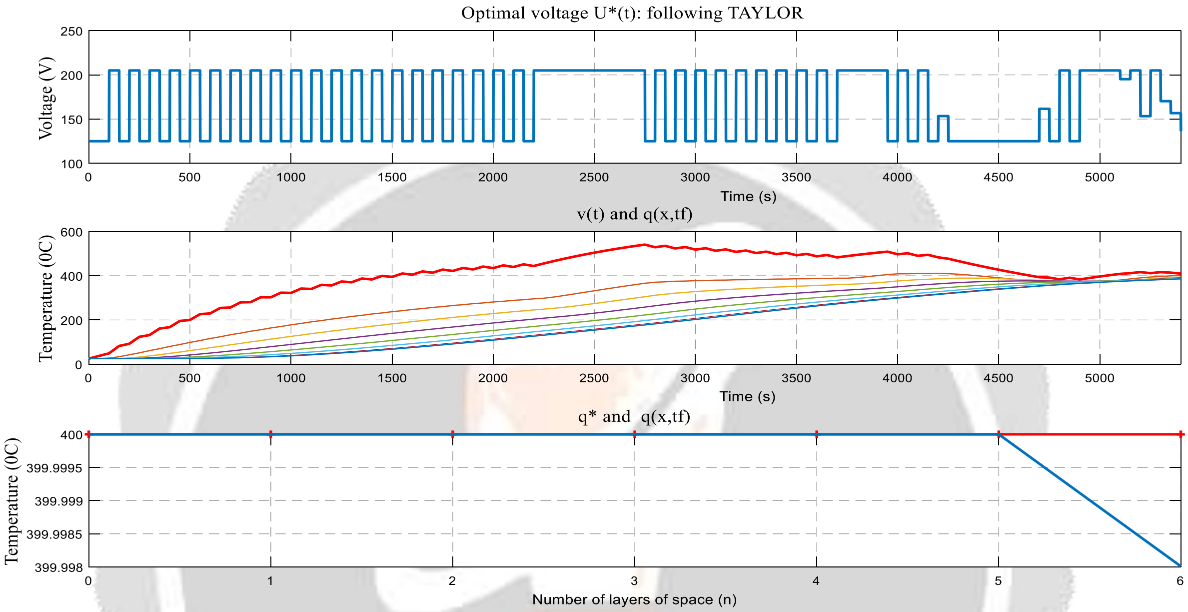

Figure 1. The optimal heating process for a flat-slab of Diatomite with $q^*= 400^0$C (e = 8.7929e-09)

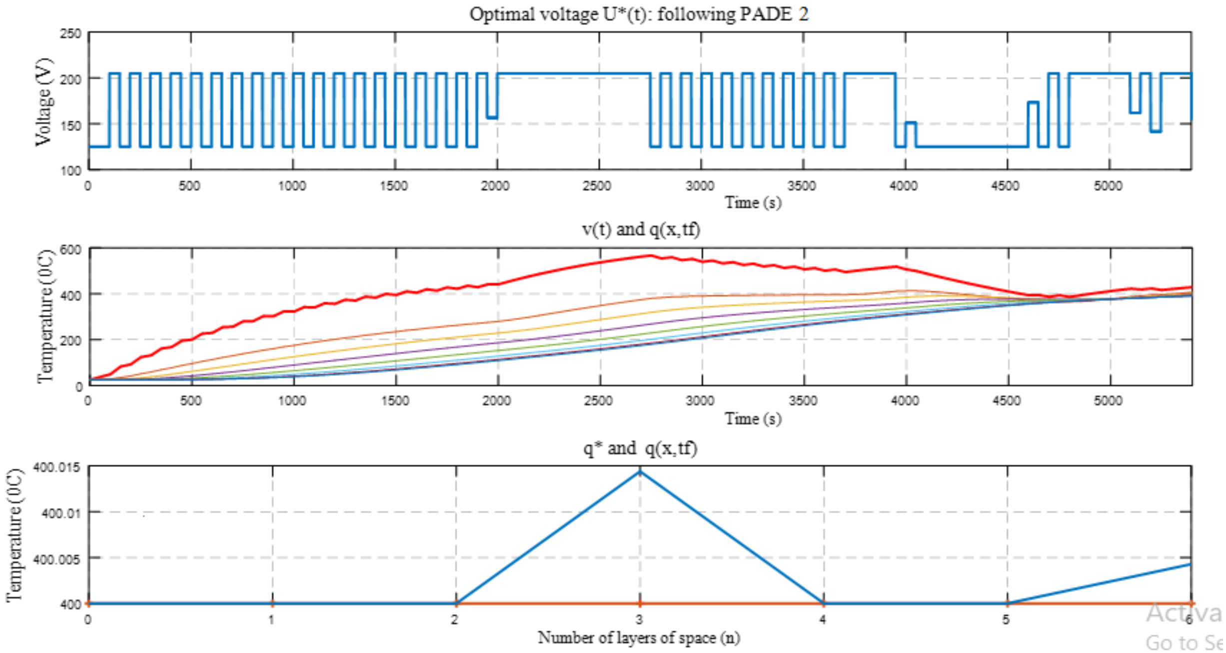

Figure 2. The optimal heating process for a flat-slab of Diatomite with $q^*= 400^0$C (e = 9.6096e-06)

4.2. Case 2: when delayed objects satisfy the condition: $2 \le T/\tau < 6$ in [4,7]

In the simulation process, we also keep all the parameters as in the case 1 but only change the delayed time τ, in this case for τ = 210 (s), so we have: $T/\tau = 2 \le 1200/210 = 5.7 < 6$. After the simulation, we also have results like in Figure 3 and Figure 4.

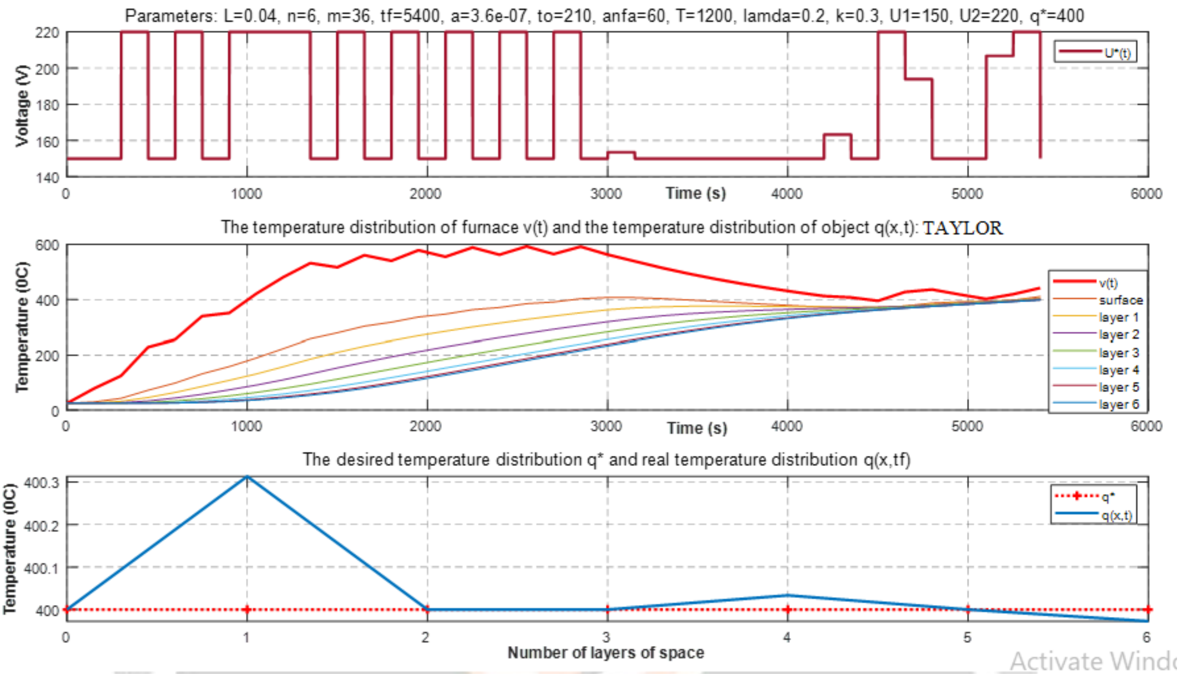

Figure 3. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C ($e \approx 0.23841$)

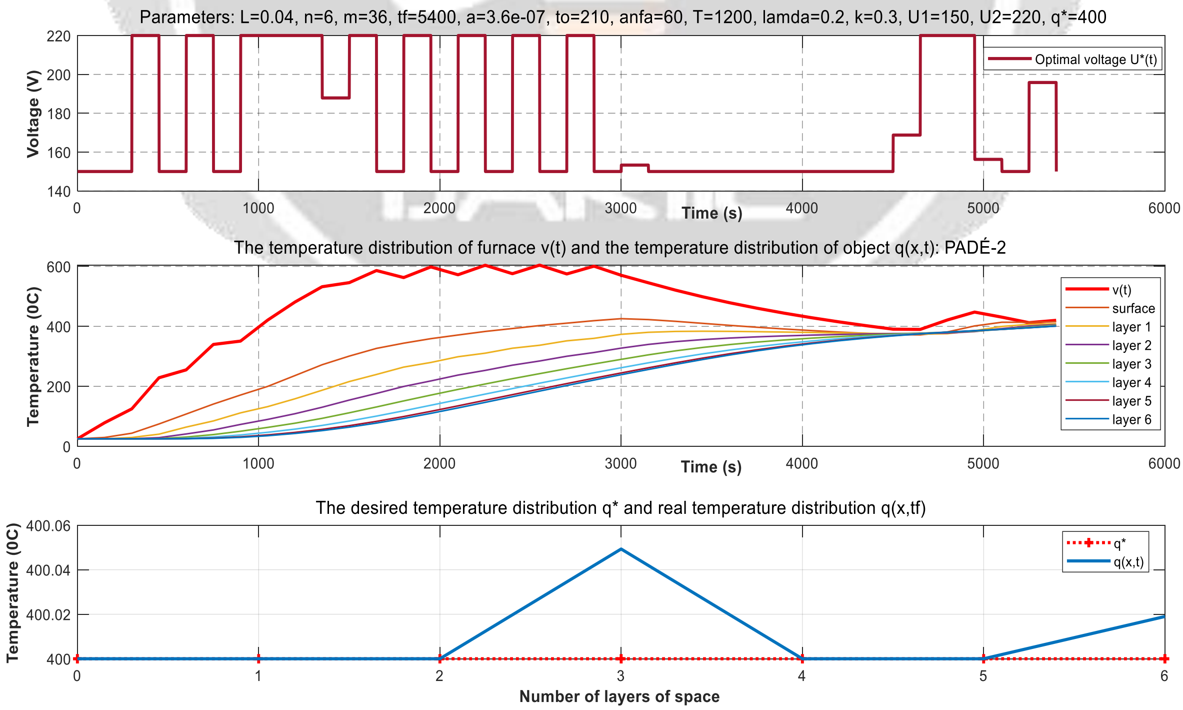

Figure 4. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C ($e \approx 0.00194$)

5. COMPARISON OF TWO METHODS

Figure 1 and Figure 2 show that both methods at the time $t=t_f$ =5400s the temperature distribution at layers $q(x,t_f)$ is approximately 400^0C, when approximating according to Taylor, the error of objective function J as $e = 8.7929e\text{-}09$ and according to Padé-2 the error of objective function J as $e = 9.6096e\text{-}06$. So when the delayed object satisfies the condition $T/\tau \geq 10$, the Taylor approximation will have smaller deviation than the Padé-2 approximation.

Figure 3 and Figure 4 also show that at the time $t=t_f$=5400s the temperature distribution at layers $q(x,t_f)$ is also approximately 400^0C, but when approximating according to Taylor, the error of objective function J as $e \approx 0.23841$ and according to Padé-2 the error of objective function J as $e \approx 0.00194$. Thus, when the delayed object satisfies the condition $2 \leq T/\tau < 6$, the Padé-2 approximation will have smaller deviation than the Taylor approximation.

6. CONCLUSIONS

The paper presented two replacing methods for a delayed object by using Taylor approximation model and Padé-2 approximation model so as to solve the optimal control problem for a distributed parameter system with delayed time. The system is applied to a specific one-sided heat-transfer system in a heating furnace to control temperature for a flat-slab following the most accurate burning standards**.** We have found an optimal voltage $u^*(t)$ so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It also means that at the end of the heating process to ensure temperature uniformity throughout the whole material.

The simulation results on a Diatomite sample have shown the correctness of the algorithms and the two methods have also shown that depending on the relationship between (τ) and (T), which approximation form is best to use.

Namely, when delayed object has (τ) and (T) satisfying $T/\tau \geq 10$, using Taylor approximation model will have higher accuracy. If delayed object satisfies the condition $2 \leq T/\tau < 6$, using Padé-2 approximation will have higher accuracy.

From the above conclusions, it can be seen that depending on the ratio T/τ, we can replace a delayed object in form of $e^{-\tau s}$ by a suitable approximation model, then the accuracy of the problem will be higher.

7. ACKNOWLEDGEMENT

The work described in this paper was supported by Thai Nguyen University of Technology (http://www.tnut.edu.vn/).

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BIOGRAPHIES

	Dr. Mai Trung Thai Lecturer at Division of Electronic Engineering, Faculty of Electronic, Thai Nguyen University of Technology. Reasearching Fields: Electronic; Control Engineering and Automation; Optimal control for a distributed parameter system.