Dispersive medium modeling using Drude model in FDTD method and Z transform

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ABSTRACT

In electromagnetics, a dispersive material is a material with electromagnetic parameters dependent on frequency. In this paper, the implementation of simulations of medium made up of plasma is studied. For this, the FDTD (Finite Difference Time Domain) method, using the DB-FDTD formulation is used. In order to modeling the plasma, the Drude model is taken into consideration in this work. The material equations, relating flux densities to fields, are described in the frequency domain and therefore must be rewritten in the time domain, for use in an FDTD formulation. In order to switch material equation's from the frequency domain to the discretized time domain, the Z transform is used. The simulation implements an FDTD grid terminated by loss layers for ABC (Absorbing Boundary Condition), and uses as source a plane wave composed with a modulated Gaussian.

Keyword: *Plasma, Drude, DB-FDTD, Z transform, Loss layers, Plane Wave, modulated Gaussian*

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1. INTRODUCTION

For many problems, results with acceptable accuracy are obtained with the FDTD method assuming that the material parameters are constants. However, constant material parameters are inherently an approximation. A non-unity, scalar, constant relative permittivity is equivalent to assuming that the charge polarization in a material is instantaneous and perfectly proportioned to the applied electric field. Usually for materials such as plasma, the permittivity is dependent of the frequency of the wave traveling through the material. When the permittivity or the permeability of an material are functions of frequency, the material is said to be dispersive. This paper focuses on the modeling of dispersive materials, and more particularly plasma, with the FDTD method.

For the numerical implementation, the DB-FDTD formulation is used. And for modeling dispersive medium, the Drude model is used. This model presents the electrical and magnetic susceptibilities as a function of the frequency.

2. DISPERSIVE MATERIALS AND CONSTITUTIVE EQUATIONS

2.1. Electrical and magnetic susceptibilities

In non-dispersive materials, the electric and magnetic flux densities are related to electric and magnetic fields via the material equations (Eq.1), whose electromagnetic parameters are given in Eq.2 [1] [2].

$$\vec{D} = \varepsilon\vec{E} \tag{1.a}$$

$$\vec{B} = \mu\vec{H} \tag{1.b}$$

where

$$\varepsilon = \varepsilon_0(\varepsilon_\infty + \chi_e) \tag{2.a}$$

$$\mu = \mu_0(\varepsilon_\infty + \chi_m) \tag{2.b}$$

In Eq.2, χ_e and $\chi_m$are the electrical and magnetic susceptibilities, and ε_0 and $\mu_0$are respectively the permittivity and the permeability of the vacuum. In dispersive materials, permittivity and permeability are frequency dependent. The material equations connecting the flux densities and the fields are thus rewritten in Eq.3. The electromagnetic parameters, dependent on the frequency, are expressed in the forms given in Eq. 4 [1][3].

$$\hat{D}(\omega) = \hat{\varepsilon}(\omega)\hat{E}(\omega) \tag{3.a}$$

$$\hat{B}(\omega) = \hat{\mu}(\omega)\hat{H}(\omega) \tag{3.b}$$

where

$$\hat{\varepsilon}(\omega) = \varepsilon_0[\varepsilon_\infty + \hat{\chi}_e(\omega)] \tag{4.a}$$

$$\hat{\mu}(\omega) = \mu_0[\mu_\infty + \hat{\chi}_m(\omega)] \tag{4.b}$$

In Eq. 4, ε_∞ and μ_∞ are respectively the permittivity and the permeability relating to the optical frequencies.

2.2. Drude model

The electrical and magnetic susceptibilities for the Drude model are given in Eq. 5 [3].

$$\hat{\chi}_{e,Dr}(\omega) = \frac{\omega_{pe}^2}{j\nu_e\omega - \omega^2} \tag{5.a}$$

$$\hat{\chi}_{m,Dr}(\omega) = \frac{\omega_{pm}^2}{j\nu_m\omega - \omega^2} \tag{5.b}$$

ω_p is the plasma frequency and ν is the electron collision frequency. In the Drude model, expenses are expected to move under the influence of the electric field, and they also undergo a damping force (Eq.6).

$$\frac{\partial^2}{\partial t^2}P = \varepsilon_0\omega_{pe}^2E(t) - \Gamma_e\frac{\partial}{\partial t}P \tag{6}$$

The model is called the plasma model. For EM waves, the plasma appears as a high pass filter. At a relatively low frequencies (frequencies lower than plasma frequency), waves are reflected. At a relatively high frequencies (frequencies higher than plasma frequency), the medium becomes transparent to these high frequency waves. More common forms of plasma susceptibilities are given in Eq. 7, where ν $(rad.s^{-1})$ represents the collision frequency of electrons.

$$\hat{\chi}_{e,P}(\omega) = \frac{\omega_{pe}^2}{j\nu_e\omega - \omega^2} \tag{7.a}$$

$$\hat{\chi}_{m,P}(\omega) = \frac{\omega_{pm}^2}{j\nu_m\omega - \omega^2} \tag{7.b}$$

3. FDTD IMPLEMENTATION OF PLASMA WITH GRID TERMINATED WITH LOSSY LAYERS

3.1. Update equations for flux densities

The implementation of a dispersive medium with the FDTD method is facilitated by using the DB-FDTD formulation. The use of loss layers can be done separately from the definition of the material (which here is a dispersive material). The electric flux density and magnetic flux density updating equations, for a DB-FDTD formulation with a grid terminated with loss layers, are given in Eq.8 and Eq.9. Eq.10 represents the calculation of the loss factors following the direction defined by d $(d = i, j, k)$ [1][4][5].

$$D_x^{n+1}(i,j,k) = C_{dd}(i,j,k)D_x^n(i,j,k) + C_{dh}(i,j,k)\left\{\left(\frac{H_z^{n+\frac{1}{2}}(i,j,k) - H_z^{n+\frac{1}{2}}(i,j-1,k)}{\Delta y}\right) - \left(\frac{H_y^{n+\frac{1}{2}}(i,j,k+1) - H_y^{n+\frac{1}{2}}(i,j,k-1)}{\Delta x}\right)\right\} \quad (8.a)$$

$$D_y^{n+1}(i,j+1,k) = C_{dd}(i,j+1,k)D_y^n(i,j+1,k) + C_{dh}(i,j+1,k)\left\{\left(\frac{H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n+\frac{1}{2}}(i,j,k-1)}{\Delta z}\right) - \left(\frac{H_z^{n+\frac{1}{2}}(i,j,k) - H_z^{n+\frac{1}{2}}(i-1,j,k)}{\Delta x}\right)\right\} \quad (8.b)$$

$$D_z^{n+1}(i,j,k+1) = C_{dd}(i,j,k+1)D_z^n(i,j,k+1) + C_{dh}(i,j,k+1)\left\{\left(\frac{H_y^{n+\frac{1}{2}}(i,j,k) - H_y^{n+\frac{1}{2}}(i-1,j,k)}{\Delta x}\right) - \left(\frac{H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n+\frac{1}{2}}(i,j-1,k)}{\Delta y}\right)\right\} \quad (8.c)$$

$$C_{dd}(i,j,k) = \frac{1 - pe(i,jk)}{1 + pe(i,jk)} \quad ; \quad C_{dh}(i,j,k) = \frac{\Delta t}{1 + pe(i,jk)} \quad (8.d)$$

$$B_x^{n+\frac{1}{2}}(i,j,k) = C_{bb}(i,j,k)B_x^{n-\frac{1}{2}}(i,j,k) + C_{be}(i,j,k)\left\{\left(\frac{E_y^n(i,j,k+1) - E_y^n(i,j,k)}{\Delta z}\right) - \left(\frac{E_z^n(i,j+1,k) - E_z^n(i,j,k)}{\Delta y}\right)\right\} \quad (9.a)$$

$$B_y^{n+\frac{1}{2}}(i,j,k) = C_{bb}(i,j,k)B_y^{n-\frac{1}{2}}(i,j,k) + C_{be}(i,j,k)\left\{\left(\frac{E_z^n(i+1,j,k) - E_z^n(i,j,k)}{\Delta x}\right) - \left(\frac{E_x^n(i,j,k+1) - E_x^n(i,j,k)}{\Delta z}\right)\right\} \quad (9.b)$$

$$B_z^{n+\frac{1}{2}}(i,j,k) = C_{bb}(i,j,k)B_z^{n-\frac{1}{2}}(i,j,k) + C_{be}(i,j,k)\left\{\left(\frac{E_x^n(i,j+1,k) - E_x^n(i,j,k)}{\Delta y}\right) - \left(\frac{E_y^n(i+1,j,k) - E_y^n(i,j,k)}{\Delta x}\right)\right\} \quad (9.c)$$

$$C_{bb}(i,j,k) = \frac{1 - p_m(i,j,k)}{1 + p_m(i,j,k)} \quad ; \quad C_{be}(i,j,k) = \frac{\Delta t}{1 + p_m(i,j,k)} \quad (9.d)$$

$$pe(d) = pm(d) = 0.333\left(\frac{d}{taille_{perte}}\right)^3;$$

$$d = [1, taille_{perte}],\ et\ d = [taille_{grille,d} - taille_{perte}, taille_{grille,d}] \tag{10}$$

3.2. Field FDTD formulation for plasma

In order to update the electric and magnetic fields, the material equations relating the flux densities and the fields are given in Eq. 11.

$$\hat{D}(\omega) = \hat{\varepsilon}(\omega).\hat{E}(\omega) \tag{11.a}$$

$$\hat{B}(\omega) = \hat{\mu}(\omega)\hat{H}(\omega) \tag{11.b}$$

3.2.1. Electric field update

The frequency dependent permittivity of the plasma is given by Eq. 12. By applying the simple element decomposition to Eq.12, Eq.13 is obtained [3].

$$\hat{\varepsilon}(\omega) = \varepsilon_0\left(\varepsilon_\infty + \frac{\omega_{pe}^2}{\omega(j\nu_e - \omega)}\right) \tag{12}$$

$$\hat{\varepsilon}(\omega) = \varepsilon_0\left(\varepsilon_\infty + \frac{\omega_{pe}^2}{\nu_e}\frac{1}{j\omega} - \frac{\omega_{pe}^2}{\nu_e}\frac{1}{\nu_e + j\omega}\right) \tag{13}$$

Eq. 13 represents the permittivity used for the following. In order to go from the frequency domain to the discretized time domain, the Z transform method will be used. The multiplication in the frequency domain corresponds to another multiplication in the Z domain. Applying the Z transformation technique, for a dispersive material, to the permittivity (Eq.13) gives Eq.14. And the Z transform of the material equation of the electric flux density (Eq.11.a) is given to Eq.15 [3].

$$\hat{\varepsilon}(Z) = \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\omega_{pe}^2}{\nu_e}\frac{1}{1-Z^{-1}} - \frac{\varepsilon_0\omega_{pe}^2}{\nu_e}\frac{1}{1-e^{-\nu_e\Delta t}Z^{-1}} \tag{14}$$

$$\hat{D}(Z) = \hat{\varepsilon}(Z).\hat{E}(Z).\Delta t \tag{15}$$

By inserting Eq.14 into Eq.15, Eq.16 is obtained. Eq.16 can be rewritten like Eq.17 in order to find a solution to $\hat{E}(Z)$. An auxiliary term I_e (Eq.18) is defined in order to reformulate Eq.17. Thus, the resolution of the electric field in the Z domain is given in Eq. 19.

$$\hat{D}(Z) = \left[\frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\omega_{pe}^2}{\nu_e}\frac{1}{1-Z^{-1}} - \frac{\varepsilon_0\omega_{pe}^2}{\nu_e}\frac{1}{1-e^{-\nu_e\Delta t}Z^{-1}}\right]\hat{E}(Z).\Delta t \tag{16}$$

$$\hat{D}(Z) = \varepsilon_0\varepsilon_\infty\hat{E}(Z) + \frac{\varepsilon_0\omega_{pe}^2\Delta t}{\nu_e}\left[\frac{(1-e^{-\nu_e\Delta t})Z^{-1}}{1-(1+e^{-\nu_e\Delta t})Z^{-1}+e^{-\nu_e\Delta t}Z^{-2}}\right]\hat{E}(Z) \tag{17}$$

$$\hat{I}_e(Z) = \frac{\varepsilon_0\omega_{pe}^2\Delta t}{\nu_e}\left[\frac{(1-e^{-\nu_e\Delta t})}{1-(1+e^{-\nu_e\Delta t})Z^{-1}+e^{-\nu_e\Delta t}Z^{-2}}\right]\hat{E}(Z) \tag{18}$$

$$\hat{E}(Z) = \frac{1}{\varepsilon_0\varepsilon_\infty}\left[\hat{D}(Z) - Z^{-1}\hat{I}_e(Z)\right] \tag{19.a}$$

$$\hat{I}_e(Z) = (1+e^{-\nu_e\Delta t})Z^{-1}\hat{I}_e(Z) - e^{-\nu_e\Delta t}Z^{-2}\hat{I}_e(Z) + \frac{\varepsilon_0\omega_{pe}^2\Delta t}{\nu_e}(1-e^{-\nu_e\Delta t})\hat{E}(Z) \tag{19.b}$$

Therefore, the formulation in the discretized time domain of the electric field, for an FDTD formulation, is given in Eq. 20.

$$E^n = \frac{1}{\varepsilon_0 \varepsilon_\infty}(D^n - I_e^{n-1}) \tag{20.a}$$

$$I_e^n = (1 + e^{-\nu_e \Delta t}) I_e^{n-1} - e^{-\nu_e \Delta t} I_e^{n-2} + \frac{\varepsilon_0 \omega_{pe}^2 \Delta t}{\nu_e}(1 - e^{-\nu_e \Delta t}) E^n \tag{20.b}$$

3.2.2. Magnetic field update

The permeability for a Drude material is given in Eq. 21. Using the same approach as for obtaining the electric field update equations, the magnetic field update equations are given in Eq. 22.

$$\hat{\mu}(\omega) = \mu_0 \left(\mu_\infty + \frac{\omega_{pm}^2}{\omega(j\nu_m - \omega)} \right) \tag{21}$$

$$H^n = \frac{1}{\mu_0 \mu_\infty}(B^n - I_m^{n-1}) \tag{22.a}$$

$$I_m^n = (1 + e^{-\nu_m \Delta t}) I_m^{n-1} - e^{-\nu_m \Delta t} I_m^{n-2} + \frac{\mu_0 \omega_{pm}^2 \Delta t}{\nu_m}(1 - e^{-\nu_m \Delta t}) H^n \tag{22.b}$$

3.3. FDTD algorithm for a Drude material

The FDTD algorithm for a Drude material is similar to the DB-FDTD algorithm which is a formulation implemented using the electric and magnetic flux densities. The only changes made are the definitions of the multiplication coefficients of the auxiliary terms (I_e and I_m). The update coefficients of the auxiliary terms are dependents of the values of the collisions frequencies of the electrons ($\nu_e(i,j,k), \nu_m(i,j,k)$) defined for each node of the grid [5].

For the vacuum, a zero value should be chosen for the collision frequency of the electrons. But referring to Eq.20.b, the update coefficient of the auxiliary term I_e contains a division by ν_e, which will give an infinite value for $\nu_e = 0$. Thus in order to circumvent this problem a very low value will be chosen for the collision frequency of the electrons ($\nu_e = 10^{-4}\ Hz$).

4. FDTD SIMULATION OF NON-MAGNETIZED PLASMA

4.1. Properties of non-magnetized plasma

The permittivity and permeability, dependent of the frequency, of the plasma are given in Eq. 23, where $\varepsilon_\infty = 1$ and $\mu_\infty = 1$ [3].

$$\hat{\varepsilon}(\omega) = \varepsilon_0 \hat{\varepsilon}_r(\omega) = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega(j\nu_e - \omega)} \right) \tag{23.a}$$

$$\hat{\mu}(\omega) = \mu_0 \hat{\mu}_r(\omega) = \mu_0 \left(1 + \frac{\omega_{pm}^2}{\omega(j\nu_m - \omega)} \right) \tag{23.b}$$

The simulation consists of a wave propagating in free space and striking the plasma. The simulation uses the properties of copper where the electron frequency collision is $\nu_e = 22.25\ THz$, and the plasma frequency is $f_{pe} = 2\ 127.83\ THz$ [6].

The conductivity of the plasma can be defined from the equation of the relative permittivity (Eq.23.a), by comparing this equation with the expression of the relative permittivity of a conductive medium (Eq.24.a). The expression of the conductivity, depending on the frequency, is therefore obtained from Eq. 24.b.

$$\hat{\varepsilon}_r(\omega) = 1 - j\frac{\hat{\sigma}(\omega)}{\omega\varepsilon_0} \quad (24.a)$$

$$\hat{\sigma}(\omega) = \frac{\varepsilon_0\omega_p^2}{\nu_e + j\omega} \quad (24.b)$$

Using Eq. 24, Fig. 1 illustrate the permittivity and conductivity of the plasma using the properties of copper. For frequencies lower than that of plasma, the medium is reflective due to its high conductivity. The higher is the frequency of the incident wave in the medium, more is the conductivity of the plasma decreases, so the medium formed by the plasma becomes transparent.

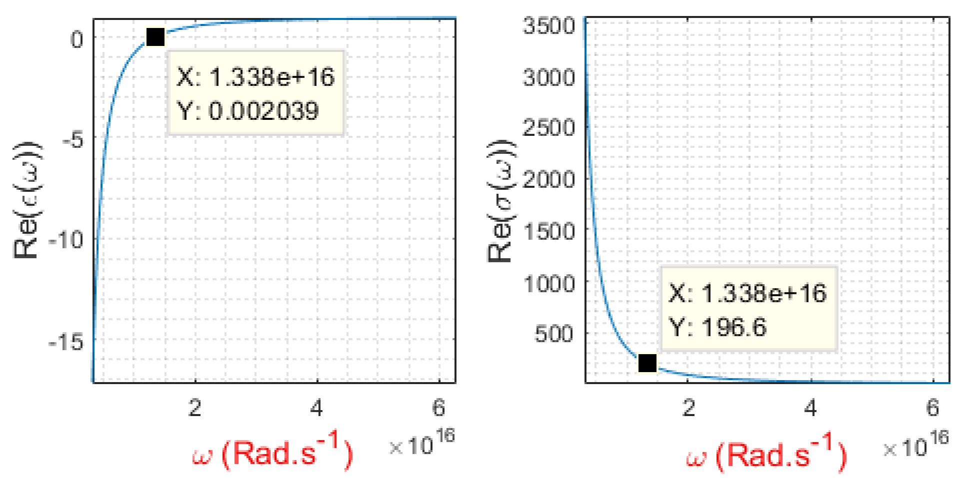


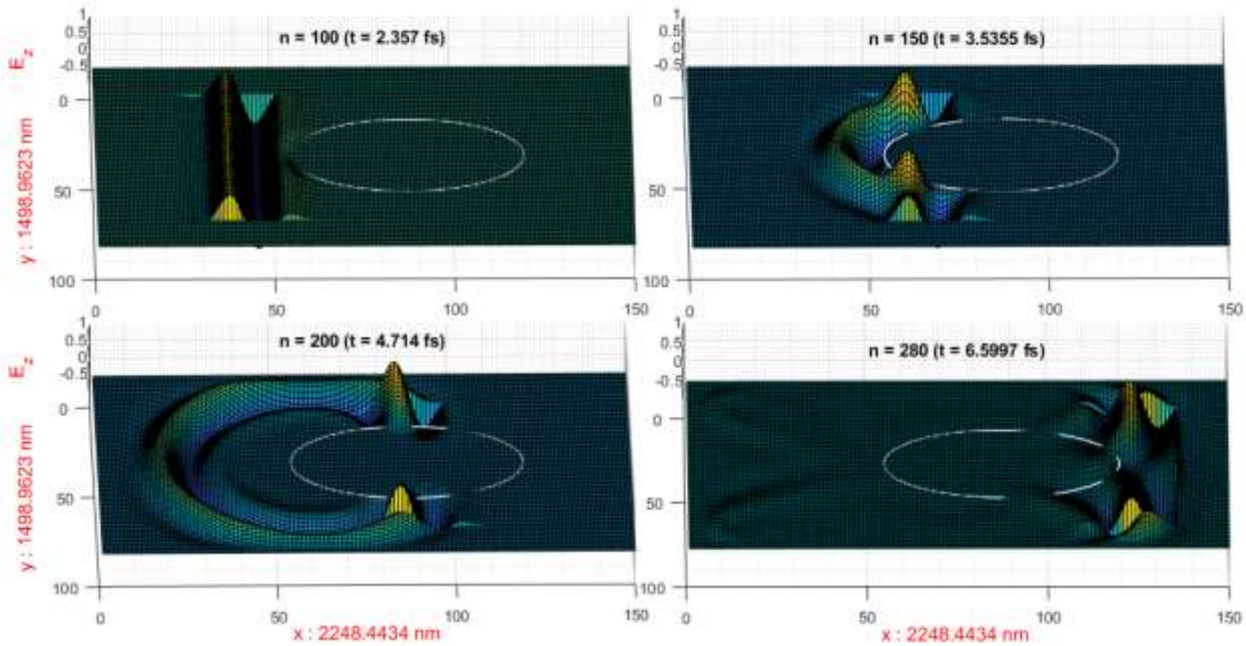
Fig.1: Real parts of the frequency dependent permittivity and conductivity for copper plasma

4.2. Simulations of a plane wave hitting a cylinder of plasma

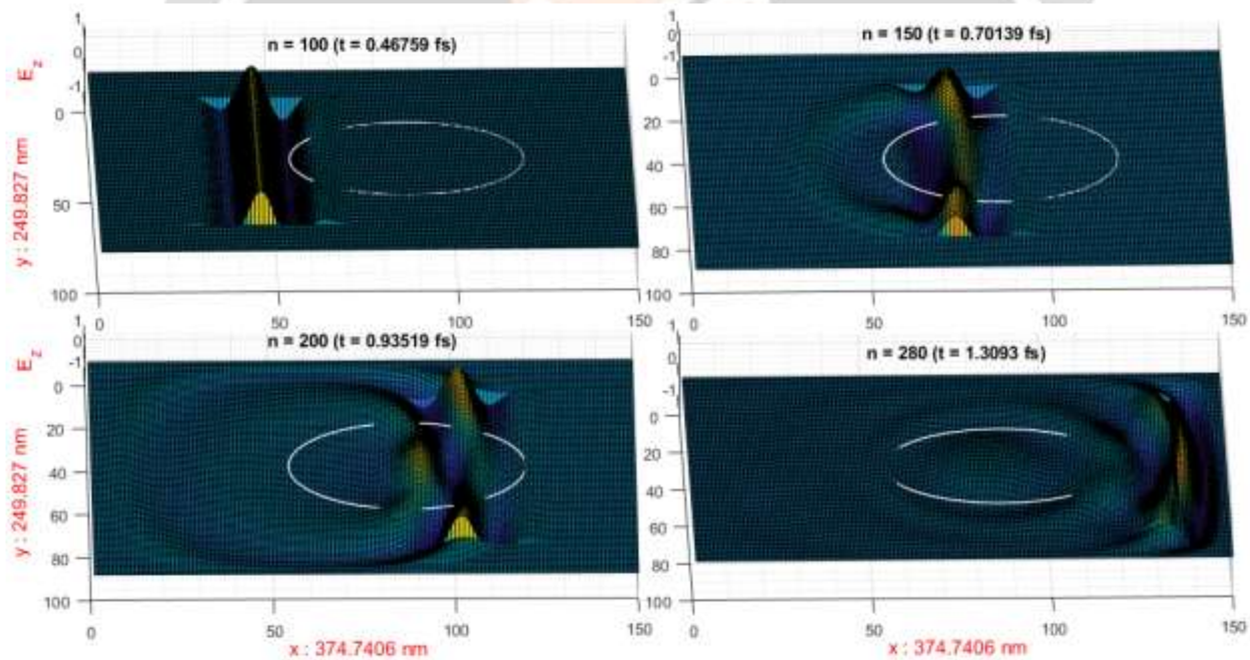
The medium composed of plasma is a cylinder of radius $R = 20 \times \Delta x$, of center c of coordinate $(75,50)$. The grid is defined with a dimension of $150 \times 100\, nodes$, and is finished with absorbent layers with $15\, nodes$ thick, at each of its limits. The number of points per wavelength is defined by $N_\lambda = 20$. The source is a modulated Gaussian, introduced by TFSF limit at node 15 of the grid.

The simulation is done in the TM mode where the components of the fields involved are E_Z for the electric field and H_x and H_y for the magnetic field. It's the same case for the components of the flux densities, D_z , B_x and B_y which are those involved in the simulation.

Fig.2, Fig.3 and Fig.4 represent the snapshots of propagation of the modulated Gaussian wave with the frequencies $1000\, THz$, $5000\, THz$ and $2127.83\, Thz$, respectively. Wave's incident at the edges of the grid are not affected by the implementation of the TFSF limit, and therefore can go outside this limit to be absorbed by the absorbent layers at the limits of the FDTD grid.



<u>Fig.2</u>: Modulated Gaussian with frequency 1000 THz striking a plasma cylinder



<u>Fig.3</u>: Modulated Gaussian with frequency 5 000 THz striking a plasma cylinder

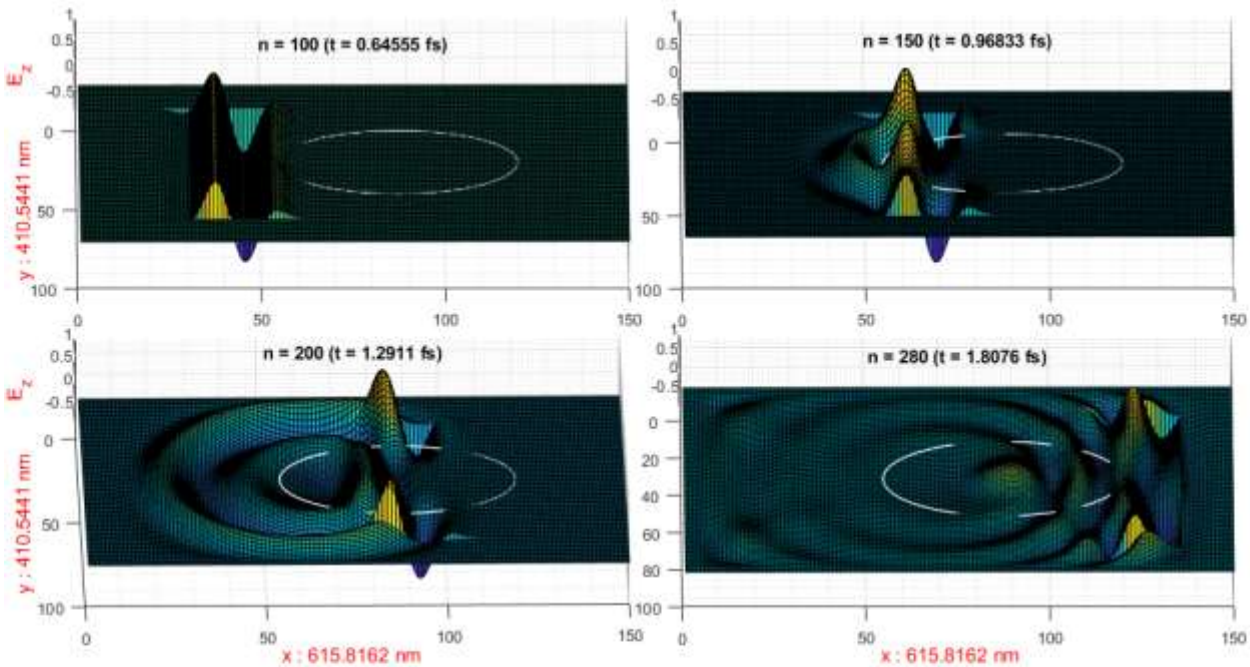


Fig.4: Modulated Gaussian with frequency 2 127.83 THz striking a plasma cylinder

5. CONCLUSION

The FDTD method makes possible the modeling of a dispersive medium such as, in the case of this work, plasma. The numerical results indicate that the behavior of the model corresponds to the physical behavior of the plasma. The FDTD method being a method working in the time domain, in order to simulate plasma it is necessary to pass the material equations from the frequency domain to the time domain. For this, the frequency domain has been passed into the discretized frequency domain using the Z transform. Using the inverse Z transform, the equations move from the Z domain to the discretized time domain. Processing of the medium was facilitated by the use of the DB-FDTD formulation. Insertion of dispersive material into the medium does not affect TFSF (for plane wave simulation) and ABC (for grid termination) formulations by using DB-FDTD formulation.

6. REFERENCES

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