Distribution of occupied resources by fuzzy customers in a shared-resource system

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ABSTRACT

In a queue with shared-resources across customers whose arrival and service parameters are fuzzy, the resource utilization distribution can also be determined as fuzzy numbers. The α-cuts of these probabilities are expressed in this paper to establish the membership functions of these fuzzy probabilities. Then, they are used for dimensioning the necessary and sufficient amount of resources required for the system so that the blocking probability does not exceed a certain threshold ε.

Keyword: - *fuzzy, queueing system, sharing, resources, utilization*

1. INTRODUCTION

The queuing theory has many applications today if we only mention traffic systems, communication systems, production systems, etc. For these application variants, the theory is based on the arrival process of customers and the service duration model they require. Although there is always a tendency to use stochastic models specifying the probability distributions of these two quantities, the parameters of the arrival and service distributions are still difficult to determine.

Fuzzy queuing theories have emerged by Li and Lee [1] to study queuing in a fuzzy environment. He posed fuzzy parameters of arrivals and services using Zadeh's extension principle [2]. Multiple angles of theories regarding fuzzy queues followed after: analyzes of fuzzy queues [3], retrial fuzzy queues [4], fuzzy queues with policy [5], multi-server queues [6], queues with priority [7], with other customers behaviors [8], using various types of fuzzy numbers: triangular or trapezoidal [9], …

For our case, we will use fuzzy number theories to model the utilizations of shared resources of queue servers. It is an extension of the model [10] that we established with a non-fuzzy queue.

2. MODEL WITH CRISP CUSTOMERS

We have established a model giving the distribution of the amount of resources used by the customers in an M/M/1 queue with a single server having discrete shared resources [10].

Given a discrete resource queue of infinite capacity. Each customer it serves will use a random amount r of resources. Since the capacity of the queue server is infinite, all customers who come to the queue are served immediately. All customers are then served simultaneously. If the system serves k customers, a priori, the probability that these k customers use an amount of resources r is denoted $P_k(r)$.

For a Poisson arrival, and an exponential service time of the customers at the queue, the probability that r $(r \geq 1)$ system resources are used is given by:

$$P(R = r) = \sum_{k=1}^{+\infty} P(N = k) . P_k(r) = \sum_{k=1}^{+\infty} \frac{\rho^k}{k!} e^{-\rho} . P_k(r) \tag{1}$$

where $P(N = k)$ denotes the probability of finding k customers served in the queue, and $\rho = \lambda/\mu$ the load factor of the queue.
The probability that the server is free ($r = 0$) is given by the probabilty that any customer is served in the queue.

This expression can be used to determine the amount of resources sufficient for a system so that the blocking rate (or the blocking probability) does not exceed a given threshold ε.

3. FUZZY SETS AND FUZZY NUMBERS

Let be Ω an infinite and uncountable set. We develop in the following few theories of sets and fuzzy numbers [11].

3.1. Fuzzy set

A fuzzy set $\tilde{E}$ on Ω is defined by its membership function $\mu_{\tilde{E}}: \Omega \to [0,1]$.

It can be characterized by:

- Its kernel which represents the elements of Ω for which the membership degree is equal to 1: $ker(\tilde{E}) = \{x \in \Omega \mid \mu_{\tilde{E}}(x) = 1\}$,
- Its support which represents the subsets of Ω for which the membership degree is non-zero: $supp(\tilde{E}) = \{x \in \Omega \mid \mu_{\tilde{E}}(x) \neq 0\}$,
- Its height which represents the maximum degree of membership in $\tilde{E}$: $h(\tilde{E}) = \max_{x\in\Omega}(\mu_{\tilde{E}}(x))$,
- Its α-cut which represents the subsets of Ω for which the membership degree is greater than or equal to α ($\alpha \in [0,1]$): $\tilde{E}[\alpha] = \{x \in \Omega \mid \mu_{\tilde{E}}(x) \geq \alpha\}$.

3.2. Fuzzy number

A fuzzy number $\tilde{N}$ is a fuzzy set satisfying the following properties:

- Its kernel is equal to a singleton: $|ker(\tilde{N})| = 1$,
- Its height is equal to 1: $h(\tilde{N}) = 1$,
- There is no disjoint α-cut: $\forall\alpha \in [0,1], \forall(u,v) \in \tilde{N}[\alpha], \forall\beta \in [0,1], \beta u + (1-\beta)v \in \tilde{N}[\alpha]$,
- Its membership function $\mu_{\tilde{N}}$ is continuous.

A triangular fuzzy number $\tilde{A}$ is a fuzzy number whose membership function is a triangular function defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & if\ a_1 < x < a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & if\ a_2 < x < a_3 \\ 0 & otherwise \end{cases} \quad (2)$$

We will note it in the form $\tilde{A} = (a_1, a_2, a_3)$ where a_1 is the lower limit of its support, a_2 its kernel, and a_3 the upper limit of its support.

Its α-cut is defined by the interval:

$$\tilde{A}[\alpha] = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha\ ] \quad (3)$$

3.3. Operations on fuzzy numbers

$\circledast$ denotes an operation (addition, or subtraction, or multiplication, or division) of two fuzzy numbers.
Consider two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ whose α-cuts are respectively denoted by $\tilde{A}[\alpha]$ and $\tilde{B}[\alpha]$.

We can define a fuzzy number $\tilde{C}$ result of the operation $\tilde{A} \circledast \tilde{B}$ such that the α-cut of $\tilde{C}$ is equal to the operation between the α-cuts of $\tilde{A}$ and $\tilde{B}$, i.e. $\tilde{A}[\alpha] \circledast \tilde{B}[\alpha] = \tilde{C}[\alpha]$. The operation on the fuzzy numbers is therefore reduced to the operation on the intervals of the α-cuts.

Given two closed intervals $[g_1, d_1]$ and $g_2, d_2]$, the operations on these intervals are defined by:

- Sum: $[g_1, d_1] + [g_2, d_2] = [g_1 + g_2, d_1 + d_2]$

- Subtraction: $[g_1, d_1] - [g_2, d_2] = [g_1 - g_2, d_1 - d_2]$
- Multiplication: $[g_1, d_1] \times [g_2, d_2] = [m_{min}, m_{max}]$
  Such that $m_{min} = \min\{g_1g_2, g_1d_2, d_1g_2, d_1d_2\}$ and $m_{max} = \max\{g_1g_2, g_1d_2, d_1g_2, d_1d_2\}$
  If g_1 and g_2 are positive, we can simplify it like $[g_1, d_1] \times [g_2, d_2] = [g_1g_2, d_1d_2]$
- Division: $[g_1, d_1]/[g_2, d_2] = [g_1, d_1] \times [\frac{1}{d_2}, \frac{1}{g_2}]$
- Multiplication with a scalar: $k.[g_1, d_1] = [k.g_1, k.d_1]$, $k > 0$.

4. MODEL WITH FUZZY CUSTOMERS

Suppose now that the arrival of customers forms a Poisson process but that the arrival rate λ is not known for sure. This arrival rate varies from λ_{min} to λ_{max} and takes an average value $\bar{\lambda}$ for a fairly long observation period T. It can be modeled as a triangular fuzzy number $\tilde{\lambda} = (\lambda_{min}, \bar{\lambda}, \lambda_{max})$. Its α-cut is equal to $\tilde{\lambda}[\alpha] = [\lambda_{min} + (\bar{\lambda} - \lambda_{min})\alpha, \lambda_{max} - (\lambda_{max} - \bar{\lambda})\alpha]$.

Similarly, for the characterization of the duration of service, it is assumed that we do not know it perfectly; only it is exponential and its mean varies from μ_{min} to μ_{max} with an average value $\bar{\mu}$. This inverse of the average duration of service can also be modeled as a triangular fuzzy number $\tilde{\mu} = (\mu_{min}, \bar{\mu}, \mu_{max})$. Its α-cut is equal to $\tilde{\mu}[\alpha] = [\mu_{min} + (\bar{\mu} - \mu_{min})\alpha, \mu_{max} - (\mu_{max} - \bar{\mu})\alpha]$.

We therefore have a queue FM/FM/1 of a single server whose characteristics of arrivals and service durations are Markovian with fuzzy parameters. The server has an infinite number of resources that it shares with its customers according to their requirements.

Then, the queue load factor is equal to $\tilde{\rho} = \tilde{\lambda}/\tilde{\mu}$. Its α-cut is defined by:

$$\tilde{\rho}[\alpha] = \left[\frac{\lambda_{min} + (\bar{\lambda} - \lambda_{min})\alpha}{\mu_{max} - (\mu_{max} - \bar{\mu})\alpha}, \frac{\lambda_{max} - (\lambda_{max} - \bar{\lambda})\alpha}{\mu_{min} + (\bar{\mu} - \mu_{min})\alpha}\right] \tag{4}$$

In the following, we will denote it by $\tilde{\rho}[\alpha] = [\rho_{min}, \rho_{max}]$.

A priori, if the system serves k customers, the probability that these k customers use an amount r of resources is denoted $P_k(r)$ where k and r denote non-fuzzy numbers. $P_k(r)$ has a representation as a non-fuzzy number.
The probability of finding k served customers in the FM/FM/1 queue is equal to the fuzzy number:

$$\tilde{P}(N = k) = \widetilde{\left(\frac{\rho^k}{k!} e^{-\rho}\right)} \tag{5}$$

Its α-cut is equal to $\left(\tilde{P}(N = k)\right)[\alpha] = \left[\frac{\rho_{min}^k}{k!} e^{-\rho_{max}}, \frac{\rho_{max}^k}{k!} e^{-\rho_{min}}\right]$.
In fact:

$\rho^k[\alpha] = [\rho_{min}^k, \rho_{max}^k]$, $\left(\frac{\rho^k}{k!}\right)[\alpha] = \left[\frac{\rho_{min}^k}{k!}, \frac{\rho_{max}^k}{k!}\right]$,

$\left(\sum_{k=0}^{+\infty} \frac{\rho^k}{k!}\right)[\alpha] = \left[\sum_{k=0}^{+\infty} \frac{\rho_{min}^k}{k!}, \sum_{k=0}^{+\infty} \frac{\rho_{max}^k}{k!}\right] = [e^{\rho_{min}}, e^{\rho_{max}}]$

$$e^{-\rho}[\alpha] = \left(\frac{1}{e^{\rho}}\right)[\alpha] = \left[\frac{1}{e^{\rho_{max}}}, \frac{1}{e^{\rho_{min}}}\right] = [e^{-\rho_{max}}, e^{-\rho_{min}}]$$

Then $\left(\frac{\rho^k}{k!} e^{-\rho}\right)[\alpha] = \left[\frac{\rho_{min}^k}{k!} e^{-\rho_{max}}, \frac{\rho_{max}^k}{k!} e^{-\rho_{min}}\right]$.

The probability that r server resources are used is therefore given by the fuzzy number:

$$\tilde{P}(R = r) = \sum_{k=1}^{+\infty} \widetilde{\frac{\rho^k}{k!} e^{-\rho}}.P_k(r) \tag{6}$$

Its α-cut is equal to $\left(\tilde{P}(R = r)\right)[\alpha] = \left[\sum_{k=1}^{+\infty} \frac{\rho_{min}^k}{k!} e^{-\rho_{max}}.P_k(r), \sum_{k=1}^{+\infty} \frac{\rho_{max}^k}{k!} e^{-\rho_{min}}.P_k(r)\right]$.

Its minimum is:

$$\sum_{k=1}^{+\infty}\frac{1}{k!}\left(\frac{\lambda_{min}+(\bar{\lambda}-\lambda_{min})\alpha}{\mu_{max}-(\mu_{max}-\bar{\mu})\alpha}\right)^k e^{-\frac{\lambda_{max}-(\lambda_{max}-\bar{\lambda})\alpha}{\mu_{min}+(\bar{\mu}-\mu_{min})\alpha}}.P_k(r) \quad (7)$$

And its maximum is:

$$\sum_{k=1}^{+\infty}\frac{1}{k!}\left(\frac{\lambda_{max}-(\lambda_{max}-\bar{\lambda})\alpha}{\mu_{min}+(\bar{\mu}-\mu_{min})\alpha}\right)^k e^{-\frac{\lambda_{min}+(\bar{\lambda}-\lambda_{min})\alpha}{\mu_{max}-(\mu_{max}-\bar{\mu})\alpha}}.P_k(r) \quad (8)$$

We can also draw the graph of the membership function as a function of P and r in this case, but it is difficult to output the analytical expression of this membership function. We can still use this graph to determine the sufficient amount r of resources to be made available in the server for a threshold probability of utilization equal to ε.

5. NUMERICAL APPLICATION

We will take the example of an infinite capacity queue server sharing its resources with customers. Those customers arrive according to a Poisson process whose rate seems uncertain, with minimum value $\lambda_{min} = 1/1.3$ s^{-1}, maximum value $\lambda_{max} = 1/1.0$ s^{-1}, and mean value $\bar{\lambda} = 1/1.2$ s^{-1}. These customers request a service of random duration following an exponential distribution with uncertain average also, of minimum value $\mu_{min} = 1/0.9$ s^{-1}, of maximum value $\mu_{max} = 1/0.7$ s^{-1}, and of mean value $\bar{\mu} = 1/0.8$ s^{-1}.

Each service requires a random amount of resources following a Poisson distribution with an average of $a = 5$ resources per customer. Resources used by different customers are independent.

We will assimilate the rate of the process of arrivals to a triangular fuzzy number $\tilde{\lambda} = (\lambda_{min}, \bar{\lambda}, \lambda_{max}) = (1/1.3, 1/1.2, 1/1.0)$, and the same for the average duration of service $\tilde{\mu} = (\mu_{min}, \bar{\mu}, \mu_{max}) = (1/0.9, 1/0.8, 1/0.7)$.

Let $\alpha \in [0,1]$. The load factor of the FM/FM/1 queue is equal to the fuzzy number $\tilde{\rho} = \tilde{\lambda}/\tilde{\mu}$ whose α-cut is equal to:

$$\tilde{\rho}[\alpha] = \left[\frac{\lambda_{min}+(\bar{\lambda}-\lambda_{min})\alpha}{\mu_{max}-(\mu_{max}-\bar{\mu})\alpha}, \frac{\lambda_{max}-(\lambda_{max}-\bar{\lambda})\alpha}{\mu_{min}+(\bar{\mu}-\mu_{min})\alpha}\right]$$

$$= \left[\frac{\frac{1}{1.3}+\left(\frac{1}{1.2}-\frac{1}{1.3}\right)\alpha}{\frac{1}{0.7}-\left(\frac{1}{0.7}-\frac{1}{0.8}\right)\alpha}, \frac{\frac{1}{1.0}-\left(\frac{1}{1.0}-\frac{1}{1.2}\right)\alpha}{\frac{1}{0.9}+\left(\frac{1}{0.8}-\frac{1}{0.9}\right)\alpha}\right] = \left[\frac{0.769+0.064\alpha}{1.429-0.179\alpha}, \frac{1-0.167\alpha}{1.111+0.139\alpha}\right] \quad (9)$$

The amount of resources used by a single customer follows the Poisson distribution with parameter $a = 5$ resources, then the amount of resources used by k customers follows the Poisson distribution with parameter $ka = 5k$ resources according to the independence of the amount of resources used by different customers. Thus, given k customers, the probability that r resources are used by k customers is equal to:

$$P_k(r) = \frac{1}{r!}(5k)^r e^{-5k} \quad (10)$$

The probability that r resources are used in the queue server is a fuzzy number whose α-cut is equal to:

$$\left(\tilde{P}(R=r)\right)[\alpha] = \left[\sum_{k=1}^{+\infty}\frac{\rho_{min}^k}{k!}e^{-\rho_{max}}.P_k(r), \sum_{k=1}^{+\infty}\frac{\rho_{max}^k}{k!}e^{-\rho_{min}}.P_k(r)\right]$$

$$= \left[\begin{array}{c}\sum_{k=1}^{+\infty}\frac{1}{k!}\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha}\right)^k e^{-\left(\frac{1-0.167\alpha}{1.111+0.139\alpha}\right)}.P_k(r), \\ \sum_{k=1}^{+\infty}\frac{1}{k!}\left(\frac{1-0.167\alpha}{1.111+0.139\alpha}\right)^k e^{-\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha}\right)}.P_k(r)\end{array}\right] \quad (11)$$

The bounds of the α-cut are:

$$\left(\tilde{P}(R=r)\right)[\alpha]_{min} = \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha}\right)^{k} e^{-\left(\frac{1-0.167\alpha}{1.111+0.139\alpha}\right)} \cdot \frac{1}{r!}(5k)^{r} e^{-5k}$$
$$= \sum_{k=1}^{+\infty} \frac{1}{k!\,r!}\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha} e^{-5}\right)^{k} e^{-\left(\frac{1-0.167\alpha}{1.111+0.139\alpha}\right)} \cdot (5k)^{r} \qquad (12)$$

And:

$$\left(\tilde{P}(R=r)\right)[\alpha]_{min} = \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\frac{1-0.167\alpha}{1.111+0.139\alpha}\right)^{k} e^{-\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha}\right)} \cdot \frac{1}{r!}(5k)^{r} e^{-5k}$$
$$= \sum_{k=1}^{+\infty} \frac{1}{k!\,r!}\left(\frac{1-0.167\alpha}{1.111+0.139\alpha} e^{-5}\right)^{k} e^{-\left(\frac{0.769+0.064\alpha}{1.429-0.179\alpha}\right)} \cdot (5k)^{r} \qquad (13)$$

From this α-cut, we can draw up the membership function of the probability of using r resources of the server.

Let us take the case of the membership function of the probability that $r = 6$ server resources are busy. Table -1 below shows the bounds of the α-cut of $\tilde{P}(R = 6)$ for different α ranging from 0 to 1.

Table -1: Bounds of the α-cut of $\tilde{P}(R = 6)$ for different α ranging

α	$\tilde{P}[\alpha]_{min}$	$\tilde{P}[\alpha]_{max}$
0.0	0.0358	0.0921
0.1	0.0376	0.0880
0.2	0.0394	0.0840
0.3	0.0414	0.0802
0.4	0.0434	0.0766
0.5	0.0455	0.0731
0.6	0.0477	0.0697
0.7	0.0500	0.0664
0.8	0.0523	0.0633
0.9	0.0548	0.0603
1.0	0.0574	0.0574

From these different bounds of the α-cut, we can draw in chart -1 the graph of the membership function of the fuzzy number $\tilde{P}(R = 6)$. We found that it is not really a triangular fuzzy number.

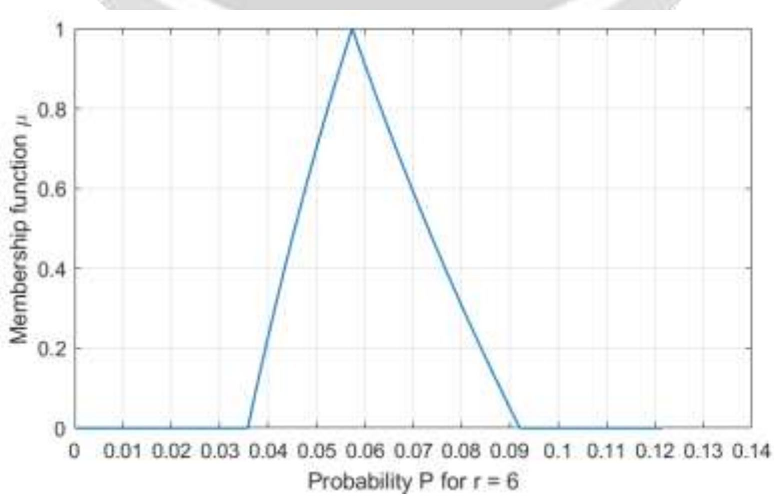

Chart -1: Membership function of $\tilde{P}(R = 6)$

The probability $\tilde{P}(R = 6)$ is a fuzzy number, whose support is equal to $[0.0358, 0.0921]$ and kernel is equal to $\{0.0574\}$.

The membership functions of the fuzzy probability of utilization of r server resources can be drawn up in the same way for different values of r. Chart -2 represents the graphs of these membership functions.

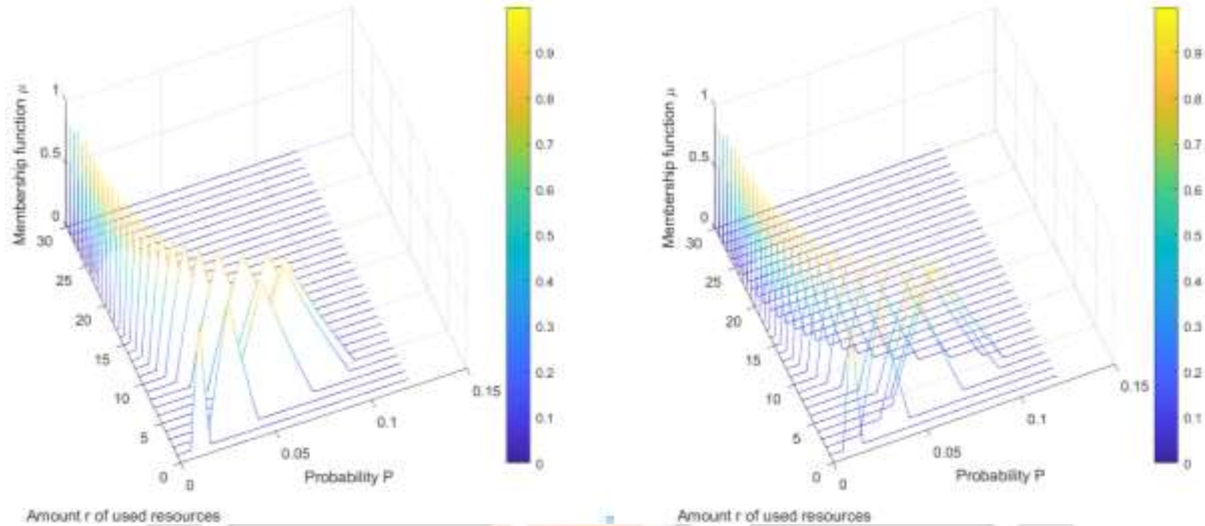

Chart -2: Membership functions of $\tilde{P}(R = r)$ for different values of r

In the chart on the left, the back curves (upper r) are hidden by the front curves (lower r) to facilitate their view. While in the chart on the right, we left them as they are.

We note the upper limit of the support of the fuzzy number $\tilde{P}$ is maximum for $r = 5$ which is equal to the average number of resources occupied by one user. The membership function curve spreads to the right (higher probability) from $r = 1$ up to $r = 5$ resources, then to the left (lower probability) from $r = 6$ resources. Although the arrival and service properties of users are fuzzy, we still note that the probability of using $r = 5$ system resources is the highest. We also found it with the analog crisp system [10].

We can also analyze this fact from the following chart -3 which represents the supports of the fuzzy numbers $\tilde{P}(R = r)$ for different values of r.

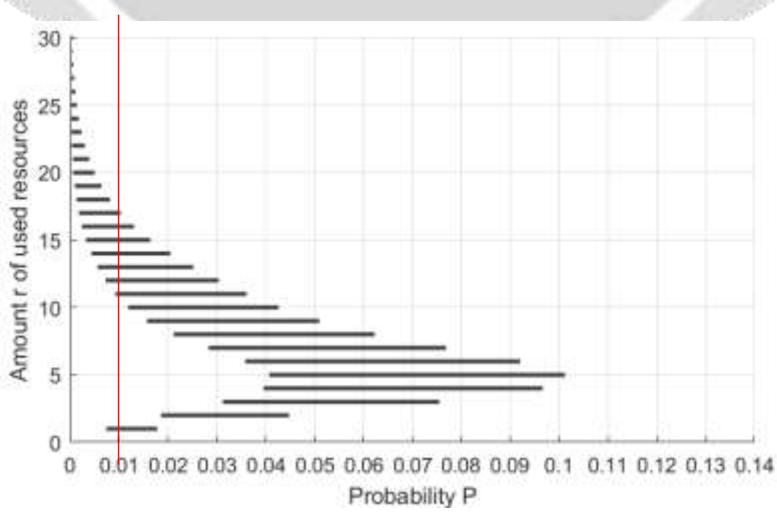

Chart -3: Support of the fuzzy probabilities $\tilde{P}(R = r)$

From this chart -3, it is possible to determine a sufficient amount of resources for the system instead of deploying an infinity which are very expensive as this support tends toward zero when the amount of resources grows up.

We now assume that the server has a finite amount C of resources. If we admit that the probability of simultaneous use of all these C resources must not exceed a threshold $\varepsilon = 0.01$, an arriving customer will find a free server resource, otherwise it must wait (blocked). In other words, the blocking probability of the system must not exceed a threshold $\varepsilon = 0.01$. Reading on chart -3, with the red limit of the probability $\varepsilon = 0.01$, we must endow the system with an amount $C \geq 18$ resources. The minimum sufficient amount is therefore 18 resources.

6. CONCLUSIONS

Fuzzy users have been called customers whose arrival and service characteristics are fuzzy parameters, i.e. comparable to fuzzy numbers. Although this information is not certain, we can estimate the amount of necessary and sufficient resources for a system from the fuzzy modeling of the distribution of occupied resources.

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