FORECASTING FUTURE POPULATION DYNAMICS IN TIRUPATI CITY: A LOGISTIC MODEL APPROACH BEYOND 2022

K.Bhanu Priya[1],Jyothi.G[2] ,M.Dhanalakshmi[3] , D.Sai Bhavani[4], Ab.Afnaan[5], N.Padmaja[6], G.Bhavani Mounika[7], G.Sumetha[8]
drjyothivellanki@gmail.com
1,2,3(Lecturers,Department of Mathematics, Sri Durga Malleswara Siddhartha Mahila Kalasala, Vijayawada, A.P, India).
4,5,6,7,8(Students, Mathematics, Sri Durga Malleswara Siddhartha Mahila Kalasala, Vijayawada, A.P, India).

ABSTRACT

This study addresses a population dynamics in Tirupati city, where the population, denoted as y, is modeled by the logistic differential equation $\frac{dy}{dx}=\frac{1}{100}y-\frac{1}{(10)^8}y^2$ *. The time, x, is measured in years, and the goal is to determine the population for* $x>2022$*. Given that the population of Tirupati city is 384,000 in the year 2022, the study aims to predict the population's behavior beyond 2022. Three specific questions were addressed:*

1. ***Population in 2023:*** *The model is used to project the population in the year 2023.*
2. ***Doubling Time:*** *The year in which the population is expected to double, based on the given differential equation, is determined.*
3. ***Ultimate Population:*** *By assuming that the differential equation applies for all* $x>2022$*, the study calculates the population's ultimate size.*

   The logistic differential equation is solved to obtain a closed-form expression for the population as a function of time. This solution is then utilized to answer the specified questions, providing insights into the future population dynamics of Tirupati city. The findings contribute to understanding the long-term trends and growth patterns, aiding in informed decision-making for urban planning and resource allocation.

Keywords: - *Population, Time, Bernoulli's Differential Equation, Malthusian law and logistic law of growth.*

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1. INTRODUCTION

POPULATION GROWTH: The population is a continuous and differential function of time. Given a population, let y be the number of individuals in it at time x. If we assume that the rate of change of the population is proportional to the number of individuals in it at any time, we are led to the differential equation $\frac{dy}{dx}=ky$ ………….(i)

Where k is a constant of proportionality. The population y is positive and is increasing and hence $\frac{dy}{dx}>0$. Therefore, from (i), we must have $k>0$. Now suppose that at time x_0 the population is y_0. Then, in addition to the differential equation (i), we have the initial condition $y(x_0)=y_0$……(ii)

The differential equation (i) is separable.

Separating the variables, integrating, and simplifying, we obtain

$y = ce^{kx}$ …………(iii)

Applying the initial condition (ii), $y = y_0 \; at \; x = x_0$, to (iii), we have

$y_0 = c\,e^{kx_0}$.

From this we at once find $c = y_0 e^{-kx_0}$ and hence obtain the unique solution

$y = y_0 e^{k(x-x_0)}$ ………….(iv) of the differential equation (i),

Which satisfies the initial condition (ii).

From (iv) we see that a population governed by the differential equation(i) with $k > 0$ and initial condition (ii) is one that increases exponentially with time. This law of population growth is called the Malthusian law.

Population growth is represented more realistically in many cases by assuming that the number of individuals y in the population at time x is described by a differential equation of the form

$\frac{dy}{dx} = ky - \lambda y^2$ ………….(v)

Where $k > 0 \; and \; \lambda > 0$ are constants. The additional term $-\lambda y^2$ is the result of some cause that tends to limit the ultimate growth of the population.

We assume that a population is described by a differential equation of the form(v) , with constants $k > 0 \; and \; \lambda > 0$, and an initial condition of the form (ii). In most such cases, it turns out that the constant λ is very small compared to the constant k. Thus for sufficiently small y, the term ky predominates, and so the population grows very rapidly for a time. However, when y becomes sufficiently large, the term $-\lambda y^2$ is of comparatively greater influence, and the result of this is a decrease in the rapid growth rate. The law of population growth described by a differential equation (v) is called the logistic law of growth. Let us consider a specific example of this type of growth.

2.STATEMENT OF THE PROBLEM: The population y of Tirupati city satisfies the logistic law $\frac{dy}{dx} = \frac{1}{100} y - \frac{1}{(10)^8} y^2$ ……..(1)where time x is measured in years. Given that the population of this city is 384,000 in 2022, determining the population as a function of time for $x > 2022$. In particular, answer the following questions.

(a) What will be the population in 2023?
(b) In what year does the 2022 population double?
(c) Assuming the differential equation(1) applies for all $x > 2022$, how large will the population ultimately be?

2.1 SOLUTION TO THE PROBLEM: we solve the differential equation(1) subject to the initial condition y(2022)= 384,000…….(2)

From(1) $\Longrightarrow \frac{dy}{dx} - \frac{1}{100} y = -\frac{1}{(10)^8} y^2$ which is a Bernoulli's Differential Equation.

Dividing with y^2 on both sides, we get

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{100}\frac{1}{y} = -\frac{1}{(10)^8} \quad \text{…………(3)}$$

Put $\frac{1}{y} = t$ ……….(4)

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx} \quad \text{………(5)}$$

Substituting (4) and (5) in (3), we get

$$-\frac{dt}{dx}-\frac{1}{100}t=-\frac{1}{(10)^8}$$

$$\frac{dt}{dx}+\frac{1}{100}t=\frac{1}{(10)^8}\quad .....(6)$$

(6) is a Linear Differential equation in 't'

Integrating factor = $e^{\int\frac{1}{100}dx}=e^{\frac{x}{100}}$

General Solution of (6) is

$$t(Integrating\ factor)=\int\frac{1}{(10)^8}\,(Integrating\ factor)\,dx+\frac{1}{k}$$

$$t\,e^{\frac{x}{100}}=\int\frac{1}{(10)^8}\,e^{\frac{x}{100}}\,dx+\frac{1}{k}$$

$$t\,e^{\frac{x}{100}}=\frac{1}{(10)^8}\int e^{\frac{x}{100}}\,dx+\frac{1}{k}$$

$$t\,e^{\frac{x}{100}}=\frac{1}{(10)^8}\,e^{\frac{x}{100}}10^2+\frac{1}{k}$$

$$t\,e^{\frac{x}{100}}=\frac{1}{(10)^6}\,e^{\frac{x}{100}}+\frac{1}{k}$$

$$t=\frac{1}{(10)^6}+\frac{1}{k}e^{-\frac{x}{100}}$$

$$\frac{1}{y}=\frac{1}{(10)^6}+\frac{1}{k}e^{-\frac{x}{100}}$$

$$\frac{1}{y}=\frac{(10)^6e^{-\frac{x}{100}}+k}{k(10)^6}$$

$$y=\frac{k(10)^6}{(10)^6e^{-\frac{x}{100}}+k}\quad ....(7)$$

(7) is the general solution of (1).

Now applying the initial condition (2) to (7), we get

$$384{,}000=\frac{k(10)^6}{(10)^6e^{-\frac{2022}{100}}+k}$$

$$[10^6\,e^{-20.22}+k]384{,}000=k10^6$$

$$384{,}000\;10^6\,e^{-20.22}+384{,}000k=k10^6$$

$$k[384{,}000-10^6]=-384{,}000\;10^6\,e^{-20.22}$$

$$k10^3[384-10^3]=-384{,}000\;10^6\,e^{-20.22}$$

$$k=\frac{-384{,}000\;10^6\;e^{-20.22}}{-616000}$$

$$k=0.001031136329$$

Substituting this value for k back into (7), we obtain the solution in the form

$$y=\frac{(0.001031136329)(10)^6}{(10)^6e^{-\frac{x}{100}}+0.001031136329}$$

$$y=\frac{1031.136329}{(10)^6e^{-\frac{x}{100}}+0.001031136329}\quad ....(8)$$

This gives the population y as a function of time for $x>2022$.

2.2 We now consider the questions (a),(b) and (c) of the problem.

Question (a) asks for the population in the year 2023.

Thus we let $x=2023\ in\ (8)$ and obtain $y=\frac{1031.136329}{(10)^6e^{-20.23}+0.001031136329}$

$$y = \frac{1031.136329}{0.002668792142}$$

$$y \approx 386{,}368$$

2.3 Question(b) asks for the year in which the population doubles. Thus we let $y = 768{,}000$ in (8) and solve for x. We have $768{,}000 = \frac{1031.136329}{(10)^6 e^{-\frac{x}{100}} + 0.001031136329}$

From which $768{,}000((10)^6 e^{-\frac{x}{100}} + 0.001031136329) = 1031.136329$

$$e^{-\frac{x}{100}} = 0.0000000003114890994$$

and hence $x \approx 2188$.

2.4 Question(c) asks how large the population will ultimately be, assuming the differential equation (1) applies for all $x > 2022$.To answer this, we evaluate lim y as $x \to \infty$ using the solution (8) of (1). We find

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1031.136329}{(10)^6 e^{-\frac{x}{100}} + 0.001031136329} = 10^6 = 1{,}000{,}000.$$

3.CONCLUSION: In this paper, the population dynamics of Tirupati city, modeled by the logistic differential equation, have been investigated. The general solution, expressed as a function of time beyond the year 2022, has been determined as $y = \frac{1031.136329}{(10)^6 e^{-\frac{x}{100}} + 0.001031136329}$. Specifically, the population in the year 2023 is estimated to be approximately 386,368. Furthermore, the year in which the population is projected to double, based on the given differential equation, is approximately 2188. Assuming the validity of the differential equation for all $x > 2022$, the ultimate population size is found to approach a limit as x tends to infinity. The ultimate population is calculated to be 1,000,000 indicating that the population is expected to stabilize at this value in the long run. This study provides valuable insights into the future population trends of Tirupati city, enabling informed decision-making for urban planning and resource allocation. The logistic model serves as a useful tool for understanding the population dynamics and making projections beyond the available data in 2022.

4.REFERENCES:

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