GENERATORS OF LORENTZ GROUP IN SIX-DIMENTIONAL SPACE-TIME

Prakash Chaudhary

Associate Professor, Department of Physics, M M H College, Ghaziabad, Uttar Pradesh.

ABSTRACT

In symmetric, psudo-Euclidean six-dimensional space-time $D(3 \oplus 3)$, six-vector covariant kinematics leads to fifteen parameter, antisymmetric, 6x6 Lorentz transformation matrix. Its sub-matrices represent relativistic kinematics and may be expressed in a new minimal 2x2 representation. Constructing infinitesimal generators of Lorentz group in six-space, their commutation relations with invariant properties are obtained. The commutations of infinitesimal generators, themselves confine in space-time representation, $[\mathrm{R}^3]$ and $[\mathrm{T}^3]$ *sub-spaces of $D(3 \oplus 3)$.*

Keyword: *Six-dimensional space-time, Extended relativity, Lorentz Group;*

1. INTRODUCTION

The need of symmetric higher dimensional space-time with minimal number of orthogonal Euclidian space-time geometry has been discussed widely in literature. The most symmetrical, minimum space-time manifold, with equal number of space and time dimensions is six-dimensional space-time $D(3 \oplus 3)$ [1-4]. In search of consistent theory of all physical phenomena in extended relativity [5-8], the multidimensional space-time theories were constructed. The kinematics, classical and quantum electrodynamics in six-dimensional space-time were studied by many authors [9-13], where temporal degrees of freedom play vital role. Through charge-field interaction in terms of Cerenkov radiation in six-space, we [14] have shown that considerable amount of energy is required to turn the time trajectory of the radiation. The experiments were also proposed to investigate higher dimensions and particle stability [9], and through space-time structural mappings [15] where physical observance of an event in higher dimensional is subjected to real, observational four-dimensional Minkowski world.

In order to construct Lorentz group in six- for complete description of the classical phenomena, In the present paper, starting with the geometry and space-time structure of six-dimensional manifold, i.e. $D(3 \oplus 3)$ space-time, the basic mathematical framework is developed in covariant six-vector structure. The three dimensional subspaces- $[R^3]$ and $[T^3] \in D(3 \oplus 3)$, inbuild in the formulation are orthogonal to each other and retain rotational invariance in respective vector space. Constructing 6x6 Special Lorentz transformation matrix its component sub-matrices are shown to preserve invariance of physical laws in six-space. The boosts in six-space may be represented in a minimal 2x2 matrix form which may incorporate internal degrees of freedom. The infinitesimal generators for Subluminal Lorentz transformations are obtained and their commutation relations are expressed in six-space. These commutation relations, themselves, confine to specific space-time rotations and hence necessarily require complete $\mathrm{D}(3 \oplus 3)$, symmetrical space-time. The temporal dimensions allow commutation relations for superluminal events in six-space.

2. STRUCTURE OF SIX-DIMENTIONAL SPACE-TIME

The six-dimensional space-time is represented by six-coordinates, with three space and three time dimensions, comprised of a spatial vector $\vec{r}$ and a temporal vector $\vec{t}$**,** such that

$$D(3 \oplus 3) \equiv (\vec{x_r}, \vec{x_t}) \qquad \text{or} \qquad (1)$$

$$D(3 \oplus 3) \equiv \{x^{\mu}\} \equiv (x_1, x_2, x_3, x_4, x_5, x_6). \quad (2)$$

These six-coordinates are orthogonal to each other, such that, a position vector $\{x^{\mu}\}$ is specified by following six-component column vector;

$$\{x^{\mu}\} \equiv \begin{bmatrix} \vec{r} \\ \vec{t} \end{bmatrix} \equiv (x^1, x^2, x^3, x^4 = t^1, x^5 = t^2, x^6 = t^3)^{\mathrm{T}} \quad (3)$$

Where T denotes the transpose. The space and time dimensions, composed of three-dimensional vector space $[R^3]$ and three-dimensional temporal vector $[T^3]$, are orthogonal to each other, such that,

$$[R^3] \perp [T^3] \quad . \quad (4)$$

In general, the velocity in six-space is a dyadic, which without loss of generality, may be expressed in six-component form [11]. We may define a unit time vector $\vec{\alpha}$, in the time field of the particle such that, the components of velocity vector $\vec{v}$ for a moving particle, are integrated with the components of the unit time vector $\vec{\alpha}$. The six-velocity vector in six-space is defined as;

$$\{ v^{\mu} \} = \mathrm{d}x^{\mu} / \mathrm{d}\tau = \gamma(v) \, [\, \vec{v}, \vec{\alpha} \,]^{\mathrm{T}} \quad (5)$$

With $\gamma(v) = \mathrm{dt} / \mathrm{d}\tau$. The time vector is directed tangentially to the time trajectory of the particle and infinitesimal increments $\overrightarrow{\mathrm{dt}}$ and $\overrightarrow{\mathrm{d}\tau}$ are measured along the time curve of the particle in moving and instantaneous frames respectively. let us consider another frame of reference K'', with relative six-velocity $\{v_{\nu}\}$ with respect to frame K'. The velocity composition law may be expressed as,

$$\Lambda\{v_{\mu}\} \; . \; \Lambda\{v_{\nu}\} = \Lambda\{v_{\nu}\} . \Lambda\{v_{\mu}\} = \Lambda\{v_{\lambda}\} \quad (6)$$

Where,

$$\{v_{\lambda}\} = [\{v_{\nu}\} + \{v_{\mu}\}] \, / \, [\, 1 + \{v_{\nu}\}^{\mathrm{T}} . \{v_{\mu}\} \,] \quad (7)$$

In general, the matrices $\Lambda\{v_{\mu}\}$ and $\Lambda\{v_{\nu}\}$ do not commute and hence Thomas precession will be predicted.

Any inertial observer, say K_o, in six space-time, may always choose the axes (t_1 , t_2 , t_3) or (t_x , t_y , t_z) or (x_4 , x_5 , x_6) in such a way that under a transcendent Lorentz transformation $\mathcal{L}$ without rotation; i.e. $\mathcal{L}$ in $\{D(3 \oplus 3)\}$, ,

$$\mathrm{x} \rightarrow t_x; \; \mathrm{y} \rightarrow t_y; \; \mathrm{z} \rightarrow t_z \quad (8)$$

and

$$t_x \rightarrow x \; ; \; t_y \rightarrow y; \; t_z \rightarrow z \quad (9)$$

The quadratic invariance between six-dimensional reference frames is represented as:

$$|ds|^2 = -|d\vec{r}|^2 + |d\vec{t}|^2 = g_{\mu\nu} x^{\mu} x^{\nu} \quad (10)$$

Where Greek indices μ□□□acquire values from 1 to 6, and $g_{\mu\nu}$ (1,1,1,-1,-1,-1) is the reference metric for $D(3 \oplus 3)$ formulation. The sub- or superscripts μ□□ =1,2,3 represent spatial coordinates and μ□□ = 4,5,6 account for temporal coordinates.

3. REPRESENTATION OF TRANSFORMATION SUB-MATRICES

An inertial observer in six-space has equal freedom to choose any of the axes, as the formal expression of LT's is independent of any space-direction. The LT's between a prime and non-prime frame K and K' in six-space [16], may be defined as,

$$\{x^{\mu}\} \rightarrow \{x^{\mu}\}' = \Lambda\{x^{\mu}\} \quad (11)$$

Where $\{x^{\mu}\}$ is six-position vector and Λ is 6x6 transformation matrix, which satisfies light speed invariance;

$$\Lambda\, g_{\mu\nu}\, \Lambda^{T} = \pm g_{\mu\nu} \quad . \tag{12}$$

We may define any sub-, or superluminal LT in six-space according to + or – sign respectively. The requirement for isotropy of the space-time leads to the following condition;

$$(\det.\Lambda)^{2} = 1 \text{ or } \quad \det.\Lambda = \pm 1 \tag{13}$$

The 6D-LT, as per positive or negative value of the determinant, has two disconnected space-time pieces- proper or non-proper. The invariance of physical laws in different inertial frames suggests that there may be two possibilities of determinant value, in each proper or nonproper, disjoint piece of SLT. Mathematically,

$$({\Lambda_{\mu}}^{\mu})^2 = 1 + \sum_{m=4}^{6} ({\Lambda_{m}}^{m})^2 \ \geq 1. \tag{14}$$

Which implies,

$$({\Lambda_{m}}^{m}) \geq +1. \quad \text{or} \quad ({\Lambda_{m}}^{m}) \leq -1 \tag{15}$$

Where m has three choices of time dimensions (= 4,5,6) in 6D space-time. Thus each subgroup has three identical, equivalent groups, associated with each choice of time dimension in $D(3\oplus 3)$ formulation. Every subspace of SLT is identical to other two subspaces and hence, it will be sufficient to study any single representative piece of SLT.

The proper, orthochronous, Subluminal LTs (SLT's) in six-space, i.e.

$$(\det.\Lambda_{\mu\nu})^{2} = 1 \text{ and } \quad ({\Lambda_{m}}^{m}) \geq +1. \tag{16}$$

are represented by The 6x6 transformation matrix Λ, which in explicit form, may be expressed as four 3x3 constituent sub-matrices A,P,Q,R -

$$\Lambda_{\mu\nu} = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \tag{17}$$

Such that, LTs may be written as,

$$\begin{Bmatrix} x^{r} \\ x^{t} \end{Bmatrix}' = \begin{bmatrix} A & P \\ Q & R \end{bmatrix} \begin{Bmatrix} x^{r} \\ x^{t} \end{Bmatrix} \tag{18}$$

The four 3x3 sub-matrices of LT satisfy following relations;

$$AA^{T} - PP^{T} = A^{T}A - Q^{T}Q = 1 \tag{19}$$

$$RR^{T} - QQ^{T} = R^{T}R - P^{T}P = 1 \tag{20}$$

$$AQ^{T} - PR^{T} = A^{T}P - Q^{T}R = 0 \tag{21}$$

The Lorentz Transformations in six-space, characterized by Λ (6x6), transformation matrix- equation(15), satisfy necessary postulates to constitute Lorentz group in 6D. Explicitly, the 3x3 sub-matrices A, P, Q, R for Lorentz boost along $\hat{z}$ (or i=3), and t = t_{α} (=4), are;

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \qquad R = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \zeta & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & 0 & \zeta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

Where γ is Lorentz factor and $\zeta = \gamma\beta$, and $\beta = v/c$. Similarly, the other special boosts may be expressed with selection of space and time indices in equation (15). In general, we have three boosts per time dimension in six-space, constituting total nine possibilities, allowing nine boosts in six-space.

The 3x3 sub-matrices A, P, Q, R may also be expressed in 2x2 matrix representation, as;

$$\overline{A} \equiv \begin{bmatrix} \hat{I} & 0 \\ 0 & \gamma \end{bmatrix}, \qquad \overline{R} \equiv \begin{bmatrix} \gamma & 0 \\ 0 & \hat{I} \end{bmatrix} \tag{24}$$

$$\overline{P} \equiv \begin{bmatrix} 0 & 0 \\ \zeta & 0 \end{bmatrix} \qquad \overline{Q} \equiv \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} \equiv [\overline{P}]^{\mathrm{T}} \tag{25}$$

Where, identity matrix $\hat{I}(2\text{x}2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (26)

The new 2x2 representation $\overline{A}$, $\overline{P}$, $\overline{Q}$, $\overline{R}$ with only one active dimension corresponding to the boosts, represents simplest form of transformation sub-matrix. The matrix relations, equation (17,18,19), are satisfied by these new 2x2 matrices and hence the invariance associated with transformations is conserved.

4. SPACE-TIME ROTATIONS IN $D(3 \oplus 3)$

In order to investigate Lie algebra of the Lorentz Group in six space, we may have three distinct possibilities of space time rotation, namely, spatial rotation in $[R^3]$, temporal rotation in $[T^3]$ and space-time rotations or boosts. To explore all possible rotations, the elements of transformation matrix are continuously connected to identity,

$$\Lambda = e^{\beta_d N_d + \theta_d M_d} \tag{27}$$

Where, β_d and θ_d are real numbers and N_d , M_d are 6x6 matrices in $D(3 \oplus 3)$. Subscript d (= r, t) is used to assign dimensionality. In equation (27), M_d represents rotation in $[R^3]$, or $[T^3]$, sub-space, and N_d represents boosts in D($3 \oplus 3$). In terms of rotation and boosts, the transformations are;

$$X' = M(\theta)\, X;\ M(\theta) = \exp(i\, \vec{\theta}.\, \vec{M}) \tag{28}$$

with $$X' = N(\beta)\, X;\quad N(\beta) = \exp(i\, \vec{\beta}.\vec{N}) \tag{29}$$

such that;

$$\vec{\theta}.\, \vec{M} = \Sigma\, \theta_d M_d \ \text{ and } \ \vec{\beta}.\, \vec{N} = \Sigma\, \beta_d N_d \qquad ; d = (\vec{r}, \vec{t}\,) \tag{30}$$

And M, N represent the generator matrices. Thus, in terms of two dimensional rotation in $[R^3]$, the components of the transformation matrix, equation (15), with μ□□ (= i,j,k) = 1,2,3 only, will be;

$$\Lambda = \begin{bmatrix} \Lambda_{jj} = \cos\theta & \Lambda_{jk} = -\sin\theta \\ \Lambda_{kj} = -\sin\theta & \Lambda_{kk} = \cos\theta \end{bmatrix} \tag{31}$$

With θ rotational angle between j and k space coordinates in cyclic order of 1,2,3. Similarly with choice of two temporal cyclic coordinates, say m and n, with μ□□ (= l,m,n) = 4,5,6 only, the transformation matrix components in temporal dimensions $[T^3]$, representing temporal rotations, will appear as,

$$\Lambda = \begin{bmatrix} \Lambda_{mm} = \cos\theta & \Lambda_{mn} = -\sin\theta \\ \Lambda_{nm} = -\sin\theta & \Lambda_{nn} = \cos\theta \end{bmatrix} \tag{32}$$

Where rotational angle θ is angle of rotation between temporal coordinates and indices m and n are m=j+3, n=k+3. In D($3 \oplus 3$), the matrix element in $[R^3]$, as well as in $[T^3]$, respectively, satisfy,

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu \end{cases} \quad (33)$$

And μ, ν ≠ j,k for equation (31) and μ, ν ≠ m,n for equation (32). The most general case of possible rotation is boost i.e. space-time rotation. If one space, say k, and one time coordinate, say m, is considered for 2D rotation, then the transformation matrix will appear as,

$$\Lambda = \begin{bmatrix} \Lambda_{kk} = -\sinh\theta & \Lambda_{mk} = -\cosh\theta \\ \Lambda_{km} = -\cosh\theta & \Lambda_{mm} = \sinh\theta \end{bmatrix} \quad (34)$$

The angle θ is the angle between space and time coordinates represented by cyclic indices k (=1,2,3) and m (=4,5,6). Thus in six-space, we have three pure rotations in $[R^3]$, three pure rotations in $[T^3]$, respectively and total nine boosts involving one-space, one-time coordinates and their permutation.

5. INFINITESIMAL GENERATORS AND THEIR COMMUTAION RELATIONS

The infinitesimal generators for rotational group in D(3 ⊕ 3), are,

$$M_{\mu\nu} = \frac{d}{d\theta} \Lambda_{\mu\nu}(\theta) \Big|_{\theta=0} \quad (35)$$

The infinitesimal generators for spatial rotation (equation 31) in respective coordinate space in $[R^3]$, are represented by,

$$M_{\mu\nu} = \begin{cases} 1 \text{ for } M_{jk} = -M_{kj} \\ 0 \text{ for } \quad \mu,\nu \neq j,k \end{cases} \quad (36)$$

The temporal rotation (equation 32) involves the same form of infinitesimal generators, in $[T^3]$, as-

$$M_{\mu\nu} = \begin{cases} 1 \text{ for } M_{mn} = -M_{nm} \\ 0 \text{ for } \quad \mu,\nu \neq m,n \end{cases} \quad (37)$$

Explicitly, out of fifteen independent matrix components, the six generators lead to specific rotations in respective spatial or temporal coordinates with components- (M_{12}, M_{23}, M_{31}), and (M_{45}, M_{56}, M_{64}). We may define specific vector forms for these generators as;

$$\overrightarrow{M^S} \equiv (M_{12}, M_{23}, M_{31}) \equiv (M_3, M_1, M_2) \quad (38)$$

$$\overrightarrow{M^T} \equiv (M_{45}, M_{56}, M_{64}) \equiv (M_6, M_4, M_5) \quad (39)$$

Where superscript S or T denotes spatial and temporal coordinate space respectively. The infinitesimal generators for space-time boosts (equation 34) in D(3 ⊕ 3), are represented by,

$$M_{\mu\nu} = \begin{cases} 1 \text{ for } M_{km} = -M_{mk} \\ 0 \text{ for } \quad \mu,\nu \neq k,m \end{cases}, \quad (40)$$

Where the $M_{\mu\nu}$ is 6x6 linearly independent matrix satisfying $M_{\mu\nu} = -M_{\nu\mu}$. The possible choice of one-space and one-time coordinate, in general, constitute nine components dyadic, involving one-space and one-time coordinate, as;

$$M_{il} = \begin{pmatrix} M_{14} & M_{24} & M_{34} \\ M_{15} & M_{25} & M_{35} \\ M_{16} & M_{26} & M_{36} \end{pmatrix}, \quad i = 1,2,3 \text{ and } l = 4,5,6. \quad (41)$$

The space-time rotation infinitesimal generators, assuming equivalence of time coordinates, may be represented in following vector form for each value of l (=4,5,6) and k=1,2,3 ;

$$\overrightarrow{N^D} \equiv (N^4)_1, (N^4)_2, (N^4)_3 \quad (42)$$

In terms of permutations, the relations for infinitesimal generators, may also be written as;

$$(\overrightarrow{M^S})_i = (1/2) \epsilon_{ijk} (M^S)^{jk} \quad ; \; (i,j,k = 1,2,3) \tag{43}$$

$$(\overrightarrow{M^T})_l = (1/2) \epsilon_{lmn} (M^T)^{mn} ; \; (l,m,n = 4,5,6) \tag{44}$$

$$\overrightarrow{N^D} \equiv (N^l)_k . \tag{45}$$

The ϵ_{ijk} and ϵ_{lmn} are permutation in spatial and temporal dimensions, respectively. The general commutation relations for the generator matrices of 6D-SLG are;

$$[(\overrightarrow{M^S})_i , (\overrightarrow{M^S})_j] = \epsilon_{ijk} \; (\overrightarrow{M^S})^k \tag{46}$$

$$[(\overrightarrow{M^T})_l , (\overrightarrow{M^T})_m] = \epsilon_{lmn} \; (\overrightarrow{M^T})^n \tag{47}$$

$$[(\overrightarrow{M^S})_i , (\overrightarrow{N^l})_j] = \epsilon_{ijk} \; (\overrightarrow{N^l})^k \tag{48}$$

$$[(\overrightarrow{N^l})_i , (\overrightarrow{N^l})_j] = - \epsilon_{ijk} \; (\overrightarrow{M^S})^k \tag{49}$$

$$[(\overrightarrow{N^l})_i , (\overrightarrow{N^m})_i] = - \epsilon_{lmn} \; (\overrightarrow{M^T})^n \tag{50}$$

$$[(\overrightarrow{M^T})_l , (\overrightarrow{N^i})_m] = \epsilon_{lmn} \; (\overrightarrow{N^i})^n \tag{51}$$

The commutations equation (46) and (47) preserve the invariance of spatial and temporal degrees of freedom and form the subalgebra isomorphic to the rotation group R(θ) in three dimensions, with condition $R^*R = R R^* = 1$ in each vector space.

6. DISCUSSION

The six-dimensional space-time represented by equation (1), with symmetric spatial and temporal degrees of freedom, has vector spaces $[R^3]$ and $[T^3] \in D(3 \oplus 3)$, which are orthogonal to each other in six space (equation 4). The quadratic invariance, equation (10), in six-space remains invariant under six-vector space-time transformation with real, non-vanishing diagonal components reference metric $g_{\mu\nu}$(1,1,1,-1,-1,-1). The relative velocity, in terms of time vector $\vec{\alpha}$ (equation 5) assumes relativistic covariant form, and 6x6 transformation matrix may be expressed in four 3x3 matrices satisfying equations (19,20,21). The relativistic effects length contraction, and time dilation in six space, under space-time structural mapping [15] lead to $M(3 \oplus 1)$, four dimensional results in Minkowski space and superluminal phenomena in $T(3 \oplus 1)$ space. The LTs constitute Lorentz group with extended structure where space-time isotropy leads to two disconnected pieces of SHG, and such disconnected piece, further divides into two pieces. The temporal degrees of freedom contribute to three-fold degeneracy corresponding to each choice of time coordinate m=4,5,6. The transformation submatrices (22), (23) also connect time dilation and length contraction as $\Delta t' = \hat{R} \, \Delta t$ and $\Delta x' = \hat{A} \, \Delta x$.

The transformation sub-matrices of $\Lambda_{\mu\nu}$ i.e. $\overline{A}, \overline{P}, \overline{Q}, \overline{R}$ represented in equations (24) and (25), carry only one-space and one-time selection, chosen for boost space-time rotation. With identity matrix equation (26), we may compactify dimensions of the matrix to lowest 2x2 representation. This representation may be used to assign internal degrees of freedom via (n x n) dimensional identity matrices. Even in four dimensions, the minimal representation, in terms of two-by-two Lorentz Group has been used to construct Wigner's little group for internal space-time symmetries [17]. The compact representation (24,25), therefore, may be used to integrate associated internal degrees of freedom.

The non-vanishing matrix components of infinitesimal generators are expressed through equations (36), (37) and (40). The vector representation of infinitesimal generators (38), (39) and (42) inherit their dependence on space-time structure. The special LTs in $M(3 \oplus 1)$, are carried by commutation relation (48) and commutation of SLT in $M(3\oplus 1)$, represented by equation (49), results in spatial vector space $[R^3]$ sub-space of $D(3\oplus 3)$. Interestingly choosing a selected time orientation through indices m, the possibility of space-dependent orientation is not abolished and becomes explicit through commutation relation (50), (51). Succession of two infinitesimal generators of SLG, equation (50) restricts the kinematics into $[T^3] \in D(3\oplus 3)$ and may be characterized as SLT in superluminal space $T(3\oplus 1)$. The commutation relations, in general seem to choose their space-time subspace, which is constituent subspace of $D(3\oplus 3)$. To complete the Lorentz algebra, the study on invariants of Homogeneous and Inhomogeneous Lorentz group in six space will be communicated shortly.

7. REFERENCES

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