INVERSE APPROXIMATION RESULT FOR MIXED SUMMATION-INTEGRAL TYPE OPERATORS

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ABSTRACT

In approximation theory, the results, which determine structural characteristics of functions from their degree of approximation, are known as inverse theorems. The study of direct and inverse theorems remains an active area of research. This research now centers on approximation by members of various non-classical and nonlinear classes such as wavelets, shift-invariant subspaces, radial basis functions, ridge functions, neural nets, multivariate splines and the like. In the present paper, we study inverse approximation property of Beta-Szasz operators in simultaneous approximation.

Key Words and Phrases: *Simultaneous approximation, Summation-integral type operators, Linear positive operators, Peetre's K-functional.*

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1. INTRODUCTION

In approximation theory, there are many-many generalizations of direct and inverse results [1, 4, 6, 7, 10, 12]. The excellent textbooks of Timan [11] and of DeVore and Lorentz [3] contain an abundance of information on direct and inverse theorems for approximation by algebraic and trigonometric polynomials. Recently, Kumar [8] proposed a new sequence of mixed summation-integral type operators and studied direct approximation results for these operators in simultaneous approximation. In the present paper we study inverse approximation estimate for these operators, which were defined as

$$B_n(f,x) = \frac{n}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} s_{n,v}(t)\,f(t)\,dt \qquad x\in[0,\infty) \tag{1.1}$$

where $f \in C_\gamma[0,\infty) \equiv \{ f \in C[0,\infty):\ f(t) \le Mt^\gamma \text{ for some } M > 0,\ \gamma > 0\}$,

$$b_{n,v}(x) = \frac{x^{v-1}}{B(n+1,v)(1+x)^{n+v+1}}\ ,\qquad s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}$$

and $B(n+1,v) = n!(v-1)!/(n+v)!$.

It is easily checked that the operators B_n are linear positive operators and it is obvious that $B_n(1,x) = 1$. Alternately the operators (1.1) may be written as

$$B_n(f,x) = \int_0^{\infty} W_n(x,t)\,f(t)\,dt,$$

where $W_n(x,t) = \dfrac{n}{n+1}\sum_{k=1}^{\infty} b_{n,v}(x)s_{n,v}(t)$.

As far as the rate of approximation is concerned the operators (1.1) are just like exponential type operators [9]. But the operators B_n are not exponential type operators, since they do not satisfy the following condition:

$$\frac{\partial}{\partial x}W_n(x,t) = \frac{n}{P(x)}W_n(x,t)(t-x), \qquad P(x) \text{ is a function of } x. \tag{1.2}$$

The above equation (1.2) is the necessary condition for the operators to be of exponential type. The above condition (1.2) is frequently used in the analysis to prove the inverse theorem for exponential type operators. In the present paper we study an inverse result in simultaneous approximation for the operators (1.1).

By C_0, we denote the class of continuous functions on the interval $(0,\infty)$ having a compact support and C_0^r is r times continuously differentiable functions with $C_0^r \subset C_0$. Suppose $G^{(r)} = \{g : g \in C_0^{r+2}, supp \quad g \subset [a',b'],\ where \quad [a',b'] \subset (a,b)\}$. For r times continuously differentiable functions f with *supp* $f \subset [a',b']$, the Peetre's K-functional are defined as

$$K_r(\xi, f, a, b) = \inf_{g\in G^{(r)}}\left[\left\|f^{(r)} - g^{(r)}\right\|_{C[a',b']} + \xi\left\{\left\|g^{(r)}\right\|_{C[a',b']} + \left\|g^{(r+2)}\right\|_{C[a',b']}\right\}\right],\ 0<\xi<1.$$

2. AUXILIARY RESULTS

This section consists of the following preliminary results, which will be helpful to prove the inverse approximation theorem in next section.

Lemma 2.1 [5]. For $m \in N \cup \{0\}$, if the m-th order moment be defined as

$$U_{n,m}(x) = \frac{1}{n+1}\sum_{k=1}^{\infty} b_{n,v}(x)\left(\frac{v-1}{n+2} - x\right)^m, \text{ then } U_{n,0}(x) = 1, U_{n,1}(x) = 0 \text{ and}$$

$$(n+2)U_{n,m+1}(x) = x(1+x)[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)].$$

Consequently, $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 2.2. Let the function $\mu_{n,m}(x), m \in N^0$, be defined as

$$\mu_{n,m}(x) = \frac{n}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} s_{n,v}(t)(t-x)^m\,dt.$$

Then $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = \dfrac{2(x+1)}{n}$ and $\mu_{n,2}(x) = \dfrac{x(x+2)n + 6(1+x)^2}{n^2}$

and there holds the recurrence relation

$$n\,\mu_{n,m+1}(x) = x(1+x)[\mu_{n,m}^{(1)}(x) + m\mu_{n,m-1}(x)] + mx\mu_{n,m-1}(x) + [m + 2(x+1)]\mu_{n,m}(x).$$

Consequently for each $x \in [0,\infty)$ we have from this recurrence relation that

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Proof. The values of $\mu_{n,0}(x), \mu_{n,1}(x)$ easily follow from the definition. We prove the recurrence relation

$$x(1+x)\mu_{n,m}^{(1)}(x)=\frac{n}{n+1}\sum_{v=1}^{\infty}x(1+x)b_{n,k}^{(1)}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^m\,dt$$

$$-\frac{mn}{n+1}\sum_{v=1}^{\infty}x(1+x)b_{n,k}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^{m-1}dt$$

Now using the identities $x(1+x)b_{n,v}^{(1)}(x)=((v-1)-(n+2)x)b_{n,v}(x)$ and $t\,s_{n,v}^{(1)}(t)=[(v-nt]s_{n,v}(t)$, we obtain

$$x(1+x)[\mu_{n,m}^{(1)}(x)+m\mu_{n,m-1}(x)]$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}(v-1-(n+2)x)b_{n,v}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}[(v-nt)+n(t-x)-(1+2x)]s_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}t\,s_{n,v}^{(1)}(t)(t-x)^m\,dt+n\,\mu_{n,\,m+1}(x)-(1+2x)\mu_{n,m}(x)$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}s_{n,v}^{(1)}(t)(t-x)^{m+1}dt+\frac{nx}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}s_{n,v}^{(1)}(t)(t-x)^m\,dt$$

$$+n\,\mu_{n,\,m+1}(x)-(1+2x)\mu_{n,m}(x).$$

$$=-(m+1)\mu_{n,m}(x)+n\mu_{n,m+1}(x)-mx\mu_{n,m-1}(x)-(1+2x)\,\mu_{n,m}(x).$$

This completes the proof of recurrence relation. The values of $\mu_{n,2}(x)$, $\mu_{n,m}(x)$ follow from the recurrence relation.

Lemma 2.3 [5]. There exist the polynomials $Q_{i,j,r}(x)$ independent of n and k such that

$$\{x(1+x)\}^r D^r[b_{n,v}(x)]=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^i(v-1-(n+2)x)^j Q_{i,j,r}(x)b_{n,v}(x),\ \text{where } D\equiv\frac{d}{dx}.$$

Lemma 2.4. Let $0<\alpha<2$ and $0<a<a'<a''<b''<b'<b<\infty$. If $f\in C_0$ with *supp* $f\subset[a'',b'']$ and $\left\|B_n^{(r)}(f,\bullet)-f^{(r)}\right\|_{C[a,b]}=O\left(n^{-\alpha/2}\right)$, then

$$K_r(\xi,f)=M_5\left\{n^{-\alpha/2}+n\xi K_r(n^{-1},f)\right\}.$$

Consequently $K_r(\xi,f)\le M_6\xi^{\alpha/2},M_2>0.$

Proof. It is sufficient to prove

$K_r(\xi,f)=M_5\left\{n^{-\alpha/2}+n\xi K_r(n^{-1},f)\right\}$, for sufficiently large *n*.

Because *supp* $f\subset[a'',b'']$, therefore by [8, Th.3.2], there exists a function $h^{(i)}\in G^{(r)},i=r,r+2$ such that

$$\left\|B_n^{(i)}(f,\bullet)-h^{(i)}\right\|_{C[a,b]}\le M_6 n^{-1}$$

Therefore

$$K_r(\xi, f) \le 3M_7 n^{-1} + \left\| B_n^{(r)}(f,\bullet) - f^{(r)} \right\|_{C[a',b']} + \xi \left\{ \left\| B_n^{(r)}(f,\bullet) \right\|_{C[a',b']} + \left\| B_n^{(r+2)}(f,\bullet) \right\|_{C[a',b']} \right\}$$

Next, it is sufficient to show that there exists a constant M_6 such that for each $g \in G^{(r)}$

$$\left\| B_n^{(r+2)}(f,\bullet) \right\|_{C[a',b']} \le M_8 n \left\{ \left\| f^{(r)} - g^{(r)} \right\|_{C[a',b']} + n^{-1} \left\| g^{(r+2)} \right\|_{C[a',b']} \right\} \tag{2.1}$$

Also using linearity property, we have

$$\left\| B_n^{(r+2)}(f,\bullet) \right\|_{C[a',b']} \le \left\| B_n^{(r+2)}(f-g,\bullet) \right\|_{C[a',b']} + \left\| B_n^{(r+2)}(g,\bullet) \right\|_{C[a',b']} \tag{2.2}$$

Applying Lemma 2.3, we get

$$\int_0^\infty \left| \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t) \right| dt \le \sum_{\substack{2i+j \le r \\ i,j \ge 0}} \sum_{k=1}^{\infty} n^i |k - nx|^j \frac{|Q_{i,j,r+2}(x)|}{\{x(1+x)\}^{r+2}} p_{n,k}(x) \int_0^\infty b_{n,k-1}(t) dt$$

$$+ (-n)(-n-1)...(-n-r-1)(1+x)^{-n-r-2}$$

Therefore by Schwarz inequality and Lemma 2.1, we obtain

$$\left\| B_n^{(r+2)}(f-g,\bullet) \right\|_{C[a',b']} \le M_9 n \left\| f^{(r)} - g^{(r)} \right\|_{C[a',b']} \tag{2.3}$$

where the constant M_9 is independent of f and g.

Next by Taylor's expansion, we have

$$g(t) = \sum_{i=0}^{r+1} \frac{g^{(i)}(x)}{i!} (t-x)^i + \frac{g^{(r+2)}(\zeta)}{(r+2)!} (t-x)^{r+2},$$

where ζ lies between t and x. Using above expansion we get

$$\left\| B_n^{(r+2)}(g,\bullet) \right\|_{C[a',b']} \le \frac{1}{(r+2)!} \left\| g^{(r+2)} \right\|_{C[a',b']} \left\| \int \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t)(t-x)^{r+2} dt \right\|_{C[a',b']} \tag{2.4}$$

Also by Lemma 2.3 and Schwarz inequality, we have

$$\left\| B_n^{(r+2)}(g,\bullet) \right\|_{C[a',b']} \le M_8 \left\| g^{(r+2)} \right\|_{C[a',b']} \tag{2.5}$$

Combining the estimates of (2.2)-(2.5), we get (2.1).
The other consequence follows from [2].
This completes the proof of the lemma.

Lemma 2.5. Let $a < a' < a'' < b'' < b' < b$ and $f^{(r)} \in C_0$ with *supp* $f \subset [a'', b'']$ then if $f \in C_0^r(\alpha, 1, a', b')$, we have $f^{(r)} \in Liz(\alpha, 1, a', b')$.

Proof. Let $|\delta| < h$ and $g \in G^{(r)}$, then for $f \in C_0^r(\alpha, 1, a', b')$

$$\left| \Delta_\delta^2 f^{(r)}(x) \right| \le \left| \Delta_\delta^2 (f^{(r)} - g^{(r)}) \right| + \left| \Delta_\delta^2 g^{(r)}(x) \right|$$

$$\le 2^2 \left\| f^{(r)} - g^{(r)} \right\|_{C[a',b']} + \delta^2 \left\| g^{(r+2)} \right\|_{C[a',b']} \le 4 M_9 K_r(\delta^2, f) \le M_{10} \delta^\alpha$$

It follows that $f \in Liz(\alpha, 1, a, b)$ i.e. $f \in Lip^*(\alpha, a, b)$.

3. INVERSE APPROXIMATION THEOREM

In this section, we shall prove our main result, namely, inverse approximation theorem, which is stated as

Theorem: Let $0<\alpha<2, 0<a_1<a_2<b_2<b_1<\infty$ and suppose $f \in C_\gamma[0,\infty)$. Then in the following statements $(i) \Rightarrow (ii)$

(i) $\left\| B_n^{(r)}(f,.) - f^{(r)} \right\|_{C[a_1,b_1]} = O\left(n^{-\alpha/2}\right)$

(ii) $f^{(r)} \in Lip^*(\alpha, a_2, b_2)$.

Proof. Let us choose a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also suppose $g \in C_0^\infty$ with *supp* $g \subset [a'', b'']$ and $g(x) = 1$ on $[a_2, b_2]$. For $x \in [a', b']$ with $D \equiv \frac{d}{dx}$, we have

$$B_n^{(r)}(fg,x) - (fg)^{(r)}(x) = D^r(B_n((fg)(t) - (fg)(x),x))$$
$$= D^r(B_n(f(t)(g(t)-g(x)),x)) + D^r(B_n(g(x)(f(t)-f(x)),x))$$
$$= J_1 + J_2 \qquad \text{(say)}$$

By Leibnitz theorem, we have

$$J_1 = \frac{\partial^r}{\partial x^r}\int_0^\infty W_n(x,t) f(t)(g(t)-g(x))dt$$
$$= \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_n^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[f(t)(g(t)-g(x))]dt$$
$$= -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x) B_n^{(i)}(f,x) + \int_0^\infty W_n^{(r)}(x,t) f(t)(g(t)-g(x))dt$$
$$= J_3 + J_4, \text{ say.}$$

Following [8, Th. 3.3], we obtain

$$J_3 = -\sum_{i=0}^{r-1}\binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) + o(n^{-\alpha/2}), \text{ uniformly in } x \in [a', b']$$

Next following [8, Th. 3.2], Schwarz inequality, Taylor's expansion of *f* and *g* and Lemma 2.2, we get

$$J_4 = \sum_{i=1}^{r}\frac{g^{(i)}(x) f^{(r-i)}(x)}{i!(r-i)!} r! + o(n^{-1/2})$$
$$= \sum_{i=1}^{r}\binom{r}{i} g^{(i)}(x) f^{(r-i)}(x) + o(n^{-\alpha/2}), \text{ uniformly in } x \in [a', b']$$

Finally applying Leibnitz theorem, we obtain

$$J_2 = \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_n^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[g(t)(f(t)-f(x))]dt$$
$$= \sum_{i=0}^{r}\binom{r}{i} g^{(r-i)}(x) B_n^{(i)}(f,x) - (fg)^{(r)}(x)$$
$$= \sum_{i=0}^{r}\binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) - (fg)^{(r)}(x) + o(n^{-\alpha/2})$$
$$= O(n^{-\alpha/2}), \text{ uniformly in } x \in [a', b'].$$

Combining the estimates of J_1, J_2, J_3 and J_4, we get

$$\left\| B_n^{(r)}(fg,.) - (fg)^{(r)} \right\|_{C[a',b']} = O\left(n^{-\alpha/2}\right).$$

Thus by Lemma 2.4 and Lemma 2.5, we have $(fg)^{(r)} \in Lip^*(\alpha, a', b')$, since $g(x) = 1$ on $[a_2, b_2]$, it follows that $f^{(r)} \in Lip^*(\alpha, a_2, b_2)$. This proves implication $(i) \Rightarrow (ii)$ for the case $0 < \alpha \le 1$.

Now to prove the implication for $1 < \alpha < 2$, for any interval $[a_1^*, b_1^*] \subset (a_1, b_1)$ and let a_2^*, b_2^* be such that $(a_2, b_2) \subset (a_2^*, b_2^*)$ and $(a_2^*, b_2^*) \subset (a_1^*, b_1^*)$. Let $\delta > 0$ we shall prove the assertion for $\alpha < 2$. From the previous case it implies that $f^{(r)}$ exists and belongs to $Lip(1-\delta, a_1^*, b_1^*)$.

Let $g \in C_0^\infty$ be such that $g(x) = 1$ on $[a_2, b_2]$ and supp $g \subset (a_2^*, b_2^*)$. Then for characteristic function $\chi_2(t)$ of the interval $[a_1^*, b_1^*]$, we have

$$\begin{aligned}\left\| B_n^{(r)}(fg,.) - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} &\le \left\| D^r [B_n(g(.)(f(t) - f(.),.) \right\|_{C[a_2^*,b_2^*]} \\ &\quad + \left\| D^r [B_n(f(t)(g(t) - g(.)),.) \right\|_{C[a_2^*,b_2^*]} \\ &= I_1 + I_2, \text{ say.}\end{aligned}$$

Following [8, Th. 3.3], we have

$$\begin{aligned} I_1 &\le \left\| D^m [B_n(g(x) f(t),.)] - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{\infty} \binom{r}{i} g^{(r-i)} B_n^{(i)}(f,.) - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)} f^{(i)} - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} + O(n^{-\alpha/2}) = O(n^{-\alpha/2}). \end{aligned}$$

Next following [8, Th. 3.2] and Leibnitz theorem, we have

$$\begin{aligned} I_2 &= \left\| - \sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)} B_n^{(i)}(f,.) + B_n^{(r)}(f(t)(g(t) - g(.))\chi_2(t),.) \right\|_{C[a_2^*,b_2^*]} \\ &= \left\| I_3 + I_4 \right\|_{C[a_2^*,b_2^*]} + O(n^{-1}) \qquad \text{(say).} \end{aligned}$$

Again following [8, Th. 3.3], we get

$$I_3 = -\sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) + O(n^{-\alpha/2}), \text{ uniformly in } x \in [a_2^*, b_2^*].$$

Applying Taylor's expansion of *f*, we have

$$\begin{aligned} I_4 &= \int_0^\infty W_n^{(r)}(x,t)[f(t)(g(t) - g(x))\chi_2(t)]dt \\ &= \sum_{i=0}^{p'+r+1} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(x,t)(t-x)^i (g(t) - g(x))\chi_2(t)dt \\ &\quad + \int_0^\infty W_n^{(r)}(x,t) \left(\frac{(f^{(r)}(\xi) - f^{(r)}(x)}{r!} \right) (t-x)^r (g(t) - g(x))\chi_2(t)dt \end{aligned}$$

(ξ lying between t and x)

$$= I_5 + I_6, \text{ say.}$$

Next following [8, Th. 3.2], we get

$$I_5 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(x,t)(t-x)^i (g(t)-g(x))dt + O(n^{-1})$$

(uniformly in $x \in [a_2^*, b_2^*]$)

$$= I_7 + O(n^{-1}) \qquad \text{(say)}$$

Since $g \in C_0^\infty$, therefore we can write

$$I_7 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{m=1}^{r+2} \frac{g^{(m)}(x)}{m!} \int_0^\infty W_n^{(r)}(x,t)(t-x)^{i+m} dt$$

$$+ \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(x,t)\varepsilon(t,x)(t-x)^{i+r+2} dt$$

(where $\varepsilon(t,x) \to 0$ as $t \to x$)

$$= I_8 + I_9 \qquad \text{(say)}$$

Following [8, Th. 3.2], we get

$$I_8 = \sum_{m=1}^{r} \frac{g^{(m)}(x)}{m!} \frac{f^{(r-m)}(x)}{(r-m)!} r! + O(n^{-1})$$

$$= \sum_{m=0}^{r} \binom{r}{m} g^{(m)}(x) f^{(r-m)}(x) + O(n^{-1}).$$

Also $I_9 = O(n^{-\alpha/2})$ uniformly in $x \in [a_2^*, b_2^*]$.

Finally using mean value theorem and Lemma 2.3, we obtain

$$\|I_6\|_{C[a_2^*,b_2^*]} \le \sum_{\substack{2m+s\le r \\ m,s\ge 0}} n^{m+s} \left\| \frac{|Q_{m,s,r}(x)|}{\{x(1+x)\}^r} \int_0^\infty W_n(x,t)|t-x|^{\delta+r+1} \times \frac{|f^{(r)}(\xi) - f^{(r)}(x)|}{r!} |g'(\eta)| \chi_2(t) dt \right\|_{C[a_2^*,b_2^*]}$$

$$= O(n^{-\delta/2})$$

where δ is chosen in such a way that $0 \le \delta \le 2-\alpha$.

Finally combining the above estimates we get

$$\left\| B_n^{(r)}(fg,.) - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} = O\left(n^{-\alpha/2}\right).$$

Since supp $fg \subset (a_2^*, b_2^*)$, it follows from Lemma 2.4 and Lemma 2.5 that $(fg)^{(r)} \in Lip^*(\alpha, a_2^*, b_2^*)$. Furthermore, since $g(x) = 1$ on $[a_2, b_2]$, we have $f^{(r)} \in Lip^*(\alpha, a_2, b_2)$.

This completes the proof of the theorem.

REFERENCES

[1] P. N. Agrawal and V. Gupta, " Inverse theorem for linear combinations of modified Bernstein polynomials, " Pure and Applied Math Sci. 40, 29-38, 1995.

[2] H. Berens and G. G. Lorentz, “ Inverse theorems for Bernstein polynomials, ” Indiana University Mathematics Journal, 21, 693-708, 1972.

[3] R. A. De-Vore and G. G. Lorentz, “ Constructive Approximation, ” Spriger-Verlag, Berlin, 1993.

[4] Z. Finta and V. Gupta, “ Direct and inverse estimates for Phillips type operators, ” J. Math Anal Appl. 303(2), 627-642, 2005.

[5] M. K. Gupta, V. Gupta and M. Kumar, “ Direct and inverse estimates for a new family of linear positive operators, ” J. Math. Anal. Appl. 330, 799-816, 2007.

[6] V. Gupta, P. Maheshwari and V. K. Jain, “ L_p-inverse theorem for modified beta operators, ” Int. J. Math. Sci., 33(20), 1395-1403, 2003.

[7] H. S. Kasana, P. N. Agrawal and V. Gupta, “ Inverse and saturation theorems for linear combination of modified Baskakov operators, ” Approx Theory and its Appl. 7 (2), 65-81, 1991.

[8] M. Kumar, “ On simultaneous approximation of functions by a certain family of linear positive operators, ” International Journal of Advanced Research in Computer Science and Software Engineering, Vol. 6, Issue 9, 452-456, 2016.

[9] C. P. May, “ Saturation and inverse theorems for combinations of a class of exponential type operators, ” Canad. J. Math. 28, 1224-1250, 1976.

[10] T. A. K. Sinha, V. Gupta, P. N. Agrawal and A. R. Gairola, “ Inverse theorem for an iterative combinations of Bernstein Durrmeyer operators, ” Studia Univ. Babes Bolyai Mathematica LIV (4), 153-164, 2009.

[11] A. F. Timan, “ Theory of Approximation of Functions of a Real Variable, ” Pergamon, Oxford, 1963.

[12] D. X. Zhou , “ L_p-inverese theorems for Beta operators, ” J. Approx. Theory, 66, 279-287, 1991.