Line Search Interior-Point Methods for Nonlinear Programming

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ABSTRACT

Interior point methods are a class of computational methods for solving optimization problem. This methods were inutility proposed by Frish [9]in 1955,Fiacco and McCormick[7]in 1968 proved global convergence for general interior point method . In this paper we concerns the formulation and analysis of these methods and present a line search primal-dual interior-point method , make some considerations about inequality constrained problems and explain the fundamental ideas involved in the present approach. Then, a basic algorithm for inequal1ty constrained optimization is presented. The numerically results about this methods is reported .

Keyword : - *constrained optimization ,nonlinear programming , primal-dual interior point methods, merit function, line search .*

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1. Introduction

In this paper we propose a primal-dual interior point method that solves nonlinearly inequality constrained optimization problems. To obtain a fast algorithm for nonlinear optimization problems using interior point methods which originally proposed during the 1960s , A large body of theory about the barrier function method was developed during the 1960s In particular, among others, Fiacco and McCormick[8] considered solving NLP problems by minimizing a sequence of smooth barrier-penalty merit functions. The inclusion of penalty terms in the merit function eliminates the equality constraints and results in unconstrained subproblems [15], The development of modern nonlinear interior-point methods was influenced by [2,4,], see also the works of Byrd, Hribar , and Nocedal [5], El-Bakry, Tapia [3] , Liu, X.W, and Sun [12] and Yamashita [17] For an analysis of interior-point methods that use filter globalization see, for example, W¨achter and Biegler [15] and The book by Nocedal, and Wright [13]and also Antonious and Sheng[1] gives a thorough presentation of several interior-point methods.

In this paper The idea is to transform the problems into a sequence of parameterized barrier function minimization problems this is presented in section 2 , In Section 3, we describe basic concepts in the primal-dual interior point method and develop the algorithm to a line search interior point methods. Section 4 reports the results of numerical experiment

2. A class of interior point methods for convex NLP

This paper consider a method for solving the optimization problem

$$\min_{x\in R^n} f(x) \qquad (1.1a)$$

$$s.t \;\; c_i(x)=0 \qquad i\in E \qquad (1,1b)$$

$$c_i(x)\ge 0 \qquad i\in I \qquad (1.1c)$$

Where x is a vector of dimension n, and functions $f : R^n \rightarrow R \ and \, c : R^n \rightarrow R^m$ are real valued and twice continuously differentiable. and E and I are finite index sets equality and inequality of constraints respectively. At any feasible set x of problem (1.1), we can define the active set A(x) to be the union of set E with the indices of the active inequality constraints at point x. the following two definitions are the linear independence constraint qualification(LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ)in nonlinear programming

Definition 1.1 (LICQ) At given point x, LICQ holds if and only if the set of active constraint gradients { $\{\nabla c_i(x) \mid i \in A(x)\}$ is linearly independent.

Definition 1.2 (MFCQ) At given point x, MFCQ holds if and only if there exists a vector d $\in R^n$ such that

$$\nabla c_i(x)^T d > 0 \qquad for\ all\ i \in A(x) \cap I$$

$$\nabla c_i(x)^T d = 0 \qquad for\ all\ i \in E$$

And the set of equality constraint gradient $\{\nabla c_i(x) \mid i \in A(x)\}$ is linealy independent.

Standerd interior point methods for 1 (see Nocedal and Wight[13].Gould, Orban and Toint [11] introduce non-negative slack s $\in R^m$

Problems with nonlinear inequality constraints can be written as equalities by adding a new variable . The slack variable s $\in R^m$ associate with all the inequality constraints, . Reformulation of (1.1a-1.1c) are

$$\min f(x) \qquad (2.1a)$$
$$s.t\ c_E(x) = 0 \quad (2.1b$$
$$c_i(x) - s = 0 \quad (2.1c)$$
$$s \geq 0.$$

We will focus on the minimization problem with inequality constraints . Equality constraints can be included in the formulation, but we ignore them to simplify the presentation. We will solve the nonlinear optimization problem by converting it into a nonlinear complementarity problem. We will present an interior point algorithm for this problem, analyze its properties and discuss conditions for polynomial complexity. Then we present a direct approach of handling nonlinear inequality constraints using barrier functions and introduce the concept of self-concordant barrier functions.

It is impossible to cover interior methods for nonlinear optimization thoroughly in anything less than a large volume. A major goal of this article is thus to show connections between classical and modern ideas and to cover highlights of both theory and practice; readers interested in learning more about interior-point methods will find an abundance of papers and books on the subject There is a vast literature for interior point methods for nonlinear optimization problem for the surveys e.g. by Forsgren, Gill, and wright[8]also Coud,Orban, and Toint[11], wachter and Biegler [14] also Byrd, Lui, and Nocedal[4] gives a presentation of several interior point methods. The class of primal-dual path-following interior point methods is considered the most successful. Mehrotra's predictor-corrector algorithm provides the basis for most implementations of this class of methods. KKT Conditions and search direction, We consider the constrained nonlinear optimization problem interior methods have advanced so far, so fast, that their influence has transformed both the theory and practice of constrained optimization Gilbert, and Nocedal [6] considered an interior-point algorithm based on the barrier function method. Yamashita and Yabe [16] considered a quasi-Newton interior-point formulation and used the Dennis-More´ theory [7] to drive a characterization of those methods.

3. INTERIOR –POINT ALGORITHM

We will focus on the minimization problem as we mention above with inequality constraints (1.1c). Equality constraints can be included, but we ignore them to simplify the presentation. The approximate problem Equations 2 is a sequence of equality constrained problems. These are easier to solve than the original inequality-constrained problem Equation 1. To solve the approximate problem, the algorithm find the steps at each iteration, A direct step is (x, s). This step attempts to solve the KKT equations, Equation 2 without the second Equation 2-1b, for the approximate problem via a linear approximation. This is also called a Newton step.

We consider two algorithm, the basic one and the second one a adopts the merit function as that of [5,6], but is based on the line search strategy. This algorithm also satisfied the condition that all the limit point are KKT point, one of the limit point is a fritz-John point and one of the limit point is an infeasible point.

Let us consider optimization problem (2.1a-2.1c) applying classical log-barrier function to this problem we obtain

$$B(x;\mu) = f(x) - \mu \sum_{i \in I} \log c_i(x)$$

Where $\mu \to 0$ is the penalty parameter. This yields the minimization problem

$$\min\ f_r(x) = f(x) - \mu \sum_{i \in I} \log s_i(x) \qquad (2.2a)$$

$$subject\ to\ c(x) - s = 0 \qquad (2.2b)$$

The Lagrangian for the problem in equation (2.2) is

$$L(x,s,\lambda,\mu) = f(x) - \mu \sum_{i=1} \log s_i - \lambda^T [c(x) - s]$$

The solution of problem satisfies the following conditions

$$\nabla f(x) - A^T(x)\lambda = 0 \qquad (3.1a)$$

$$-\mu s^{-1} e + \lambda = 0 \qquad (3.1b)$$

$$c_i(x) - s = 0 \qquad (3.1c)$$

Which are the KKT conditions. Here A (x) is the Jacobin matrix of the function c(x), and λ is the Lagrange multiplier .We define S as the diagonal matrix whose diagonal entry is given by the vectors s, and let e = (1, 1, . . . , 1)T. multiplying the equation (3.1b) by s we obtain the primal-dual system

$$\nabla f(x) - A^T(x)\lambda = 0 \qquad (4.1a)$$

$$-\mu e + S\Lambda e = 0 \qquad (4.1b)$$

$$c_i(x) - s = 0 \qquad (4.1c)$$

Where Λ is the diagonal entry (λ1, λ2, λ3,…, λq)

Appling Newton's method [7, 14, 16].to the system (4.1a-4.1c) leads to the following linear system for Newton direction

$$\begin{bmatrix} \nabla^2_{xx} l & 0 & -A_i^T(x) \\ 0 & \Lambda & -s \\ A_i(x) & -I & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_s \\ p_\lambda \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A_i^T(x) z \\ s\Lambda e - \mu e \\ c_i(x) - s \end{bmatrix} \quad (5)$$

Where $\nabla^2_{xx} l$ is the Hessian of the Lagrangian of the problem (2). The vector (px,ps,pλ)must be determined also we can find one step of the interior-point method (IPM) algorithm $(x, s, \lambda) \rightarrow (x^T, s^T, \lambda^T,)$ is as follows

$$x^+ = x + \alpha_s p_x \ , \ s^+ = s + \alpha_s p_s \quad (6.1a)$$

$$\lambda^+ = \lambda + \alpha_z p_\lambda \ , \quad (6.1b)$$

Where α is a step length chosen as

$$\alpha_s = \max\{\alpha \in (0,1] : s + \alpha p_s \geq (1-\tau)s\} \quad (7.1a)$$

with τ ∈ (0, 1).To make the matrix in (5) symmetric multiply the equation by –I and the second equation by S^{-1} we will obtained

$$\begin{bmatrix} \nabla^2_{xx} l & 0 & -A_i^T(x) \\ 0 & S^{-1}\Lambda & I \\ A_i(x) & -I & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_s \\ p_\lambda \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A_i^T(x) \lambda \\ \Lambda - \mu S^{-1} e \\ c_i(x) - s \end{bmatrix} \quad (8)$$

Let

$$\sigma = \nabla f\ (x) - A_i^T(x)\lambda \quad (9.1)$$

$$\gamma = \Lambda - \mu S^{-1} e \quad (9.2)$$

$$\rho = c(x) - s \quad (9.3)$$

The system (7) become

$$\begin{bmatrix} \nabla^2_{xx} l & 0 & -A_i^T(x) \\ 0 & S^{-1}\Lambda & I \\ A_i(x) & -I & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_s \\ p_\lambda \end{bmatrix} = \begin{bmatrix} \sigma \\ -\gamma \\ \rho \end{bmatrix} \quad (10)$$

With $\rho \in (0,1)$. By solving equation (9.2) for ps, we get

$$P_s = S\Lambda^{-1}(-\gamma - p_\lambda) \quad (11)$$

The system (8) is reduced to

$$\begin{bmatrix} \nabla^2_{xx} l & -A_i^T(x) \\ A_i(x) & S\Lambda^{-1} \end{bmatrix} \begin{bmatrix} p_x \\ p_\lambda \end{bmatrix} = \begin{bmatrix} \sigma \\ \rho + S\Lambda^{-1}\gamma \end{bmatrix} \quad (12)$$

we can used numerical linear algebra technology developed in [13] to obtain the solution of (9).To control the convergence we use the merit function of [13]

$$\phi_{\nu,\tau}(x,s) = f_\tau(x) - \mu \sum_{i=1} \log s_i + \upsilon \| c(x) - s \| \quad (13)$$

This merit function is differentiable with respect to the elements of x and s . The penalty parameter $\upsilon > 0$ can be updated by using the strategies described in [1,13].In a line search method, after the step p has been computed and the maximum step lengths $\alpha_s = \max\{\alpha \in (0,1] : s + \alpha p_s \geq (1-\tau)s\}$ have been determined, we perform a backtracking line search that computes the step lengths $\alpha_s \in (0, \alpha_s^{\max}]$ providing sufficient decrease of the merit function

$$E(x,s,y;\mu) = \max\{\|\nabla f(x) - A_E(x)^T y - A_i(x)^T y\|\} \quad (14)$$

Algorithm 1

Given a starting point $x^0 \in R^n$ *and* $s^0 \in R^n$ that strictly satisfied the constraints i.e. $x^0 > 0$, initially slack variable s0 and compute the multipliers y0 and z0 >0 select an initial barrier parameter μ_0 >0 and parameters $\sigma, \tau \in (0,1)$.initialize k:=0

Repeat until a stopping test for the problem (1.1) is satisfied

Repeat until $E(x_k, s_k, y_k, \mu_k) \leq \mu_k$

Compute the search direction (p_x,p_s,p_λ)which satisfies the linearized constraints;

Compute α_s

Compute $(x_{k+1}, s_{k+1}, y_{k+1})$

Set $\mu_{k+1} \leftarrow \mu_k$ and k=k+1

End

Choose $\mu_k \in (0, \sigma\mu_k)$;

End

The algorithm 1.1 can converge to an infeasible point if the algorithm started from any point and the algorithm many not find the stationary point . hence we introduce a new interior point method as in [13,14]. The new algorithm behaves well in terms of global convergence and it can always find a point with certain stationary properties, also some assumptions must be considered of implementation that guarantee the good behavior of line search interior point method. To control the convergence we identify some essential factors of line search interior point method ; these factors are an appropriate merit function, a suitable control procedure of slack vector and a proper steplength. Here we adopts the same merit function as in [13,14] these function is defined as follows

$$\phi\left(x,s;p\right)= f\left(x\right)-\mu\sum_{i=1}\log s_i + \rho\|c(x)-s\| \qquad (14)$$

Where ρ>0 is a penalty parameter update automatically in each iterations the function (14) is differentiable , let the derivative of this function along the P_k be

$D\phi(x,s,p_k)$ and let

$$\pi_k\left(p_k;\rho\right)= g_k^T p_x - \mu e^T S_k^{-1} p_s - \rho\left(\|c_k - s_k\|\right) - \|c_k - s_k + A_k^T p_x + p_s\| \qquad (15)$$

Where $p=(p_x,p_s)$, $g_k=\nabla f\left(x_k\right)$ and as in [13,14]

$D\phi(x,s,p;\rho)\le \pi_k\left(p_k;\rho\right)$ which less than zero i.e. p_k is a descent direction of the merit function. The linesearch interior-point algorithm for convex programming problems.

We now give the second Algorithm with the series of modification of algorithm 1

3.1LINE SEARCH INTERIOR-POINT METHOD

We now give a more detailed description of a line search interior-point method. We denote by $D\varphi(x,s;p)$ the directional derivative of the merit function φat (x,s) in the direction p.

The interior-point algorithm for convex programming problems can now be summarized as in [13,14]

Algorithm.2 .

Choose *the initial set* $x0$ and $s0 > 0$, and compute initial values for the multiplier $\lambda > 0$. Select an initial barrier parameter$\mu > 0$, parameters $\xi,\delta \in (0,1)$, $\sigma_0 < 1/2$, select β_1 *and* β_2 so that

$\beta_1 < 1 < \beta_2$ *and* $\beta_1 \mu e < s_0\lambda_0 \le \beta_2 \mu 2$

Set $k \leftarrow 0$.

Compute the primal-dual direction $p = (p_x , p_s)$

If $\pi_k\left(p_k;\rho_k\right)\le \dfrac{-1}{2} p_x^T \nabla^2 L(x,\lambda) p_x^T - \dfrac{1}{2} p_s S_k^{-1}\Lambda_k p_s \qquad (16)$

let $\rho_{k+1}=\rho_k$,update ρ_k to ρ_{k+1} such that 16 holds

Compute $\alpha_s^{\max} = \max\{\alpha \in (0,1]: s+\alpha p_s \ge (1-\tau)s\}; \tau = 0.995$

Compute step lengths αs, satisfying both $s_k + \alpha_k p_s \ge \xi s_k$ and

$\varphi (x_k + \alpha_s p_x , s_k + \alpha_s p_s;\rho_{k+1}) \le \varphi (x_k , s_k;\rho_{k+1}) + \sigma_0\delta_j\alpha_k D\varphi (x_k , s_k ; p_{k+1})$;

increase j until there exists a scalar $\gamma \in [0,1]$satisfying $\beta_1 \mu e \le \left(S_k + \delta_j \alpha_k P_s\right)\left(\lambda_k + \gamma P_\lambda\right) \le \beta_2 \mu e \quad (17)$

where P_s is diagonal matrix of p_s

for the fixed j, let γ be the max $\gamma \in [0,1]$satisfying (17) ,compute $\alpha^{\max}$

Compute $(x_{k+1}, s_{k+1}, \lambda_{k+1})$ using the equations

$X_{k+1}=x_k+\alpha_k p_x$,$s_{k+1}=sk+\alpha_k p_s$ $\lambda_{k+1}=\lambda_{k+1} =\lambda_k+{}^{\gamma}{}_k p_\lambda$

if the stopping criterion holds, stop ;(The stopping conditions are based on the error function E)

update the approximation *Bk* ;

Set $k \leftarrow k + 1$;

End.

This framework is fairly general and covers many of the current methods that use line searches (e.g. [3, 4, 9,10,11,17]). The strategy that updates the barrier parameter μ at every step is easily implemented in this framework. If the merit function can cause the Maratos effect ,a second-order correction or a nonmonotone strategy should be implemented. An alternative to using a merit function is to employ a filter mechanism to perform the line search.

4.NUMERICAL EXPERIMENTS CONCLUSIONS

We develop a matlab interior point code that solves nonlinear programming problems with constraints , following the framework of Byrd,Hribar and Nocedal [13] the example taken from [1] with the same parameter value of $\beta_1 = 0.01$, $\beta_2 = 10$, $\rho_0 = 0.1$ $\delta = 0.8$ and the intial x0=-4,s0=λ0=(1,1)T

The optimal solution is (-3.5108,0.0028e-9)

5. CONCLUDING REMARKS

An interior –point algorithm for solution of constrained problems has been proposed and analyzed

6. ACKNOWLEDGEMENT

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