MATHEMATICAL REPRESENTATIONS OF ELECTRICAL POWER

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ABSTRACT

The paper examines mathematical representations of electrical magnitudes in a.c. circuit theory. It gives an historical and technical perspective of the development of the power concept and its geometrical and algebraic interpretations. The paper criticises the existing mathematical model of electrical power for being an entanglement of two mutually inconsistent, algebras: 1) standard vector algebra (Gibbs- Heaviside) and 2) complex algebra .The paper examines the ubiquitous expressions for power: S= P+jQ; $\dot{S} = \dot{V}I^*$ The paper analyzes Steinmetz's symbolic method and exposes its inconsistencies. The paper proves that Steinmetz hypothesis, of a new and noncommutative algebra for power theory, represents a rediscovery of Grassmann- Clifford Algebra. The paper proposes a new didactic of power theory that should include Geometric Algebra.

Keyword : - *Steinmetz's symbolic method, power theory, mathematical representation of voltage, current, and power, complex algebra, standard vector algebra, Geometric Algebra. etc….*

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1. INTRODUCTION

In dealing with electrical power we behave like Heaviside: we pay for active power, we compensate for reactive power, we design considering apparent power and, we calculate power flows as complex numbers, whereas voltages and currents are vectors. However, we are not quite sure what the right mathematical representation for electrical power is: 1) vector, 2) complex number, 3) complex vector, 4) multivector in Geometric Algebra. In a sense, the lack of interest for this question is a result of our success: for more than a century the power industry continued to thrive. Even if we do not quite understand the concept and the mathematical representation of electrical power, we were not particularly concerned. The paper raises the question: should we and does it matter?
It matters because:
a.The mathematical aspects of power theory (epistemology of power phenomena) have an impact on the physical understanding of power phenomena (ontology of power phenomena) or, paraphrasing Einstein, nothing is more practical than a good theory.
b.The symbolic method is fraught with mathematical inconsistencies such as: the symbol j is both an operand and an operator; it squares to −1 (interpreted as an imaginary scalar) and, it squares to +1 (interpreted as an unit vector).
c.The didactic of power theory is a confusing knot of two mutually - inconsistent algebras.

2. THE DEFINITION OF ELECTRICAL POWER

There are two commonly used expressions for electrical power:
1) Steinmetz's expression

$P = P^1 + \mathrm{j}P^{\mathrm{j}} \equiv P + \mathrm{j}Q$

2) the expression introduced by La Cour, Bragstadt, and Janet:

$\dot{S} = \dot{V}I^*$

2.1 Steinmetz's symbolic method and power expression

Steinmetz developed his symbolic method during the period: 1890 - 1899 . Steinmetz mentions Blakesley and Kapp as predecessors in using vector symbolism. He stated that electrical magnitudes could be represented either geometrically (as vectors in polar coordinates) or algebraically (as complex numbers). Steinmetz's representation of electrical magnitudes, algebraic as complex numbers and geometrically as vectors, although widely used during his time, blurs the conceptual difference (with respect to multiplication) between the commutative algebra (ℂ-Algebra) and Vector Calculus (which is neither commutative nor noncommutative). From a strict mathematically point of view, vector calculus is not an Algebra.Steinmetz interprets electrical power as a wave of double frequency (as compared to the single frequency waves of voltage and current); he states that, complex algebra represents, , the highest level of algebra and the best suited one for a.c. circuits theory.
He gives a new interpretation of the imaginary symbol:
"Since $\mathrm{j}^2 = -1$, that is 180° rotation for $\dot{E}$ and $\dot{I}$, for the double frequency vector P, $\mathrm{j}^2 = +1$, or 360°" Steinmetz infers that, for power phenomena, the imaginary j squares like a unit vector $y\rightarrow$ and $\mathrm{j}^2 = +1$; this contradicts the basic tenet of complex algebra, i.e. $\mathrm{j}^2 = -1$.
Steinmetz introduces a ground-breaking hypothesis:
"The double frequency vector product $P = [\dot{E}\ \dot{I}]$ brings us outside the limits of algebra…and the commutative principle of algebra: $a \times b = b \times a$ does not apply any more …$[\dot{E}\ \dot{I}]$ unlike $[\dot{I}\ \dot{E}]$".
Based on the above conjectures, Steinmetz derives the expression of electrical power in rectangular coordinates
$P = [\dot{E}\ \dot{I}] = [EI]^1 + \mathrm{j}[EI]^{\mathrm{j}} = (e'i' + e''i'') + \mathrm{j}(e''i' - e'i'') = P^1 + \mathrm{j}P^{\mathrm{j}} \equiv P + \mathrm{j}Q$.
Steinmetz's books published in German [6] and French [7] ensured the wide diffusion of the symbolic method in Europe. In all his publications, he *never* used the mathematical expression: $\dot{S} = \dot{V}I^*$; it is the author's opinion that *this expression should not to be confused with Steinmetz's expression for power which is:* $P = [\dot{E}\ \dot{I}]$. Steinmetz's power expression corresponds to the canonical expression: $p = ni$.

2.2 The expression $\dot{S} = \dot{V}I^*$ - an axiomatic definition proposed by La Cour, Bragstad, and Janet.
Steinmetz's symbolic method sparked a lengthy debate (in which Steinmetz did not participate) between the proponents and opponents of complex numbers and vectors. Guilbert and Breisig proposed, in order to obtain the desired expression of electrical power, to change the sign of the imaginary:
$\mathrm{j} \rightarrow -\mathrm{j}$. In 1913, La Cour and Bragstad proposed the expression $\dot{V}\ I^*$ which was supported, as "a practical" rule, by Janet .
This expression: $\dot{S} = \dot{V}I^*$ is an axiomatic definition and should be written as:
$\dot{S} \coloneqq \dot{V}\ I^*$
The electrical power is, by definition, a product of the complex voltage and of the complex conjugate current. This expression **does not** correspond to the canonical expression: $p = ni$.

3. STEINMETZ'S LEGACY

Despite its inconsistencies, Steinmetz's symbolic method is nevertheless, because of its efficiency, widely used. Likewise the expression $\dot{S} = \dot{V}I^*$ remains the widely used expression for apparent power. The resilience of the symbolic method and of the ubiquitous expression for apparent power is based on Steinmetz's huge scientific authority and the support received from peers such as Kennelly, Janet, La Cour, etc.
However, Steinmetz's method was never been subjected to the scrutiny of a scientific debate and he never disputed viewpoints or critical observations made by Punga , Emde , Natalis , Nichols , Franklin , and Whitehead. His only debate was with Macfarlane on the issue of complex algebra's superiority versus other algebras; the history of mathematics and the development of Algebra proved that Macfarlane's position was right. Though often cited,

Steinmetz's symbolic method remains "symbolic" and in name only; people pay respect to his name while using complex numbers and vectors and phasors and…, but few bother, nowadays, to question its theoretical assumptions. The debate related to Steinmetz's symbolic method and/or representations of electrical magnitudes, missed Steinmetz's profound message, i.e. his hypothesis that: the power expression can be obtained with the help of a new noncommutative algebra. And indeed, Steinmetz's equations can be reformulated in the noncommutative algebra of Grassmann and Clifford: Geometric Algebra.

Like a Russian doll, the symbolic method is built upon four nested riddles: 1) electrical magnitudes are represented by vectors; 2) electrical magnitudes are represented by complex numbers (complexified vectors), 3) the imaginary number j squares to plus one, and nevertheless 4) the power expression corresponds to the canonic expression: $p = ni$!

How is it possible to have a flawed mathematical foundation under a heuristically successful method?

This is possible because, at the foundation of the symbolic method is neither vector algebra nor complex algebra; at the foundation of the symbolic method as algorithm for a.c. circuit calculations lays a Grassmann-Clifford eometric Algebra; although hidden under the vestments of complex and vector algebras, the symbolic method is a Geometric Algebra in disguise. In the following section, the author proves that Steinmetz's hypothesis is in fact his rediscovery of Geometric Algebra.

4.STEINMETZ'S HYPOTHESIS

Steinmetz's conjectures that, at the foundation of power expression,

$P = [\dot{E}\ \dot{I}] = [EI]^1 + \mathrm{j}[EI]^{\mathrm{j}}$

$= (e' i' + e'' i'')$

$+ \mathrm{j}(e'' i' - e' i'') \equiv P + \mathrm{j}Q$

lays a non-commutative algebra.

If we consider voltage and current as grade-1 multivectors in a 2-dimensional vector spaces, and $\bar{e_1}$, $\bar{e_2}$ as unit vectors, the voltage and current can be represented as:

$\bar{E} = e^1 \bar{e_1} + e^{11}\ \bar{e_2}$

$\bar{I} = i^1 \bar{e_1} + i^{11} \bar{e_2}$

The power expression is the geometric product of voltage and current multivectors:

$\bar{E}\bar{I} = \bar{E} \cdot \bar{I} + \bar{E}\mathrm{K}\bar{I}$

$P = e^1 i^1 + e^{11} i^{11}$

$QJ = (e^1 i^{11} - e^{11} i^1) e_{12}$

$J = e_{12}$

The expression: $e^1 i^1 + e^{11} i^{11}$ is identical to Steinmetz's expression for active power: $[P]^1$.

The expression: $e^1 i^{11} - e^{11} i^1$ is identical to Steinmetz's expression for wattless or reactive power: $[P]^{\mathrm{j}}$.

The expression: $J = e_{12}$ represents the pseudoscalar; although it squares to minus one it is mathematically different from the imaginary scalar j.

5. CONCLUSIONS

The paper demonstrates that using the GA, instead of complex algebra and standard vector algebra, we obtained the same mathematical expressions for power as Steinmetz through his symbolic method. The new paradigm based on Geometric Algebra contains elements of the algebras used in the symbolic method; complex algebra is a sub algebra of GA. Gibbs' vectors are replaced by multi vectors. Essential in the new paradigm is the concept of geometric product which can be generalized beyond 3-D. The paper removes major mathematical inconsistencies, clarifies long existing confusions and demonstrates that Steinmetz symbolic method is neither based on complex algebra nor on Gibbs-Heaviside standard vector algebra. The paper proposes a new paradigm for power theory in which, electrical magnitudes are interpreted as elements of a graded Clifford algebra. Active power is interpreted as a scalar, or a zero-graded multi vector; voltage and current are interpreted as 1-graded multi vectors and reactive power as a 2-graded multi vector or a bivector. The apparent power is a linear combination of a scalar and of a bivector. Based on these findings, the author argues for a revision of the power engineering didactic focused on the mathematical representations of electrical magnitudes in a.c. theory and power theory, i.e. epistemology of power phenomena. The paper foes not address problems related to the ontology of power phenomena.

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