NECESSARY CONDITION ON A SOLUTION FOR THE STEINER TREE PROBLEM WITH RECTILINEAR DISTANCE

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ABSTRACT

The Steiner tree problem is one of the oldest topic of mathematical and it is attentioned of reseachers. This paper is concerned with the following type of Steiner 's problem: "Given n points in the plane find the shortest tree(s) whose vertices contain these n points". Usually, the roads are straight – line connections and the distance between two points is the Euclidean distance. In this paper, however, the rectilinear distance is used. Rectilinear distance has application in printed circuit technology where n electrically commonpoints must be connected with the shortest possible length of wire and the wires must run in the horizontal and verticaldirections. Several necessary conditions are given for any n.

Keyword: *rectilinear distance, enclosing rectangle, additional vertice, grid, transferre point.*

1. INTRODUCTION

The Steiner problem is a problem that is used a lot in practice such as construction: "For the city, build a transport network with the shortest total length so that a tourist can go from city to city. another street. The road may go outside the city limits and there are intersections called junctions. The intersections were added to reduce the total length of the traffic network. ". Or application in circuit technology: " Give points on a power board connected by power lines so that the total length is shortest".

However, in the second example we see that the power lines in the table can only run horizontally or vertically, this is an example of the Steiner problem with rectangular distances. Therefore, to solve the above practical problems, in this paper, we consider the Steiner problem in terms of rectangular distance. In other words, the lines connecting the given points in the problem are the perpendicular folds going vertically and horizontally. From there, give some conclusions about the solution of the problem in this case.

2. PRELIMINARIES

Difinition 1. The rectilinear distance $d(\mathrm{p}_1,\mathrm{p}_2)$ between two points p_1 and p_2 is difined as:

$$d(\mathrm{p}_1;\mathrm{p}_2)=|x_1-x_2|+|y_1-y_2|$$

where $\mathrm{p}_1(x_1,\mathrm{y}_1),\mathrm{p}_2(x_2,y_2)$

Difinition 2. The enclosing rectangle: Given n points in the plane the enclosing rectangle is the smallest rectangle whose sides are parallel to the Ox and Oy and which includes then points either within or on its boundary.

Difinition 3. In the plane, two points (or two vertex) $p_i;p_j$ are adjacent if they have an edge in common. We let $C(p_j)$ be the set of vertices adjacent to p_i .

Difinition 4. The local degree of the vertex p_i : We let $w(p_i)$ be the local degree of the vertex p_i, that is, the number of vertices adjacent to p_i .

Difinition 5. The inner rectangle R_1: Give n points in the plane $p_i(x_i, y_i);\ i=1,...,n$ where $\{x_i\}$ and $\{y_i\}$ in increasing order. Then by drawing lines parallel to the y – axis through $x_2, x_3, ..., x_{n-1}$ and lines parallel to the x – axis through $y_2, y_3, ..., y_{n-1}$, this defines, in general, k points which we call $c_1, c_2, ..., c_k$ $(k \le (n-2)^2)$. The rectangle which has these k points at its corners is called the inner rectangle R_1.

Difinition 6. Give n points in the plane $p_i\ (i=1,...,n)$ with the inner rectangle R_1. Consider the quadrants U_{c_i} , exterior to R_1, formed by extended lines of R_1 and each of the c_1. If there is a points p_j in a quadrant U_{c_i}, then we say that p_j is transferred to the point c_1.

To solve the Steiner problem, we give two problems P_n, T_n have been solved.

P_n **:** Given n points $p_1, p_2, ..., \mathrm{p}_n$ in the plane, find a points q such that the sum of the distances from q to $p_i, i=1,2,...,n$, is minimum.

T_n : Given n points in the plane, find the shortest tree whose vertices are these n points.

3. NECESSARY CONDITIONS ON A SOLUTION FOR STEINER PROBLEM

3.1 The Steiner problem with 3 points

Theorem 3.1. In the plane, given 3 points p_1, p_2, p_3, let $(x_i, y_i), \mathrm{i}=1,2,3$ be the coordinates of them. The q – point of P_3 is located at (x_m, y_m) where x_m and y_m are the medians of $\{x_i\}$ and $\{y_i\}$, respectively.

Let $d_{S_3}, d_{P_3}, d_{T_3}$ be the total rectilinear distance in the solutions of S3, P3, T3, respectively.

Theorem 3.2. The solutions of S_3 and P_3 are indentical, in fact

$$d_{S_3} = d_{P_3} = \frac{1}{2}P(R) \le d_{T_3}$$

Where P(R) is the perimeter of the enclosing rectange. The equality sign holds only if exits $q \equiv p_i$, $(x_m, y_m) = (x_i, y_i)$ for some $i=1,2,3.$

Proof

Case 1: Three points p_1, p_2, p_3 are collinear points, respectively.

By Theorem 3.1, q point is in the straight line connecting those points and p_1, p_3 is at vertex of rectangle R.

Because $d_{S_3} = d_{P_3} = \frac{1}{2}P(R)$,

$$\sum_{i=1}^{3}(|x_m - x_i| + |y_m - y_i|) = |x_3 - x_1| + |y_3 - y_1|$$

p_1, p_2, p_3, q are in the straight line then $x_m = x_2, y_m = y_2$. Hence p_2 is the median of the segment connecting p_1, p_2 and $q \equiv p_2$. The solutions of S_3 and P_3 are indentical, the minimum tree solution to S_3 have zero additional vertex; in fact,

$$d_{S_3} = d_{P_3} = \frac{1}{2}P(R) = d_{T_3}$$

Case 2: Three points p_1, p_2, p_3 aren't collinear points.

For $d_{S_3} = d_{P_3}$, $d_{S_3} = \sum_{i=1}^{3} d(q, p_i)$ with q is the median point of the triangle $p_1p_2p_3$ (by theorem 3.1). Hence q is vertex in the minimum tree solution to S_3.

Since $d_{p_3} = \frac{1}{2}P(R)$ there must exist two points p_i (i=1,2,3) lie on the boundary of R and q- point lies inside R.

Without loss of generality, assume that p_1 lies at vertex of R.

$$\begin{aligned} d_{S_3} &= |x_1 - x_q| + |x_2 - x_q| + |x_3 - x_2| + |y_1 - y_q| + |y_2 - y_q| + |y_3 - y_2| \\ &= \frac{1}{2}P(R) \\ &= |x_2 - x_1| + |y_3 - y_1| \end{aligned}$$

Therefore q is additional vertex in the minimum tree solution to S_3.

This compeletes the proof of theorem 3.2. The fact that the minimum tree solution to S_3 can have either zero or one additional vexter. See Fig 1

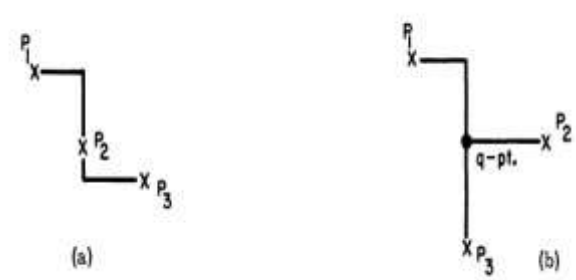


Fig. 1

3.2. Nescessary conditions on a solution to Sn

We use some notations:

p_i are the given n points (i =1,…,n), are called p – vertices and q_i, i= 1,…,k, are the additional k vertices in the solution G of S_n, are q – vertices in the solution G of S_n. We let:

$$P = \{p_i, i = 1,...,\mathrm{n}\}; \mathrm{Q} = \{\mathrm{q}_i, i = 1,...,\mathrm{k}\}$$

When we speak of a vertex a_i, i = 1,…,n+k, we mean either p_i or q_i. We let $C(a_i)$ be the set of vertices adjacent to a_i, $w(a_i)$ be the local degree of the vertex a_i, that is, the number of $C(a_i)$.

Nescessary conditions on a solution to Sn:

i) $w(q_i) = 3$ or 4 , $1 \le i \le k$
ii) $1 \le \mathrm{w}(p_i) \le 4 \;\; 1 \le i \le n$
iii) $0 \le k \le n-2$

Poof.

Condition (i) $w(q_i) = 3$ or 4 , $1 \le i \le k$

The first, we proof $w(q_i) = 3$

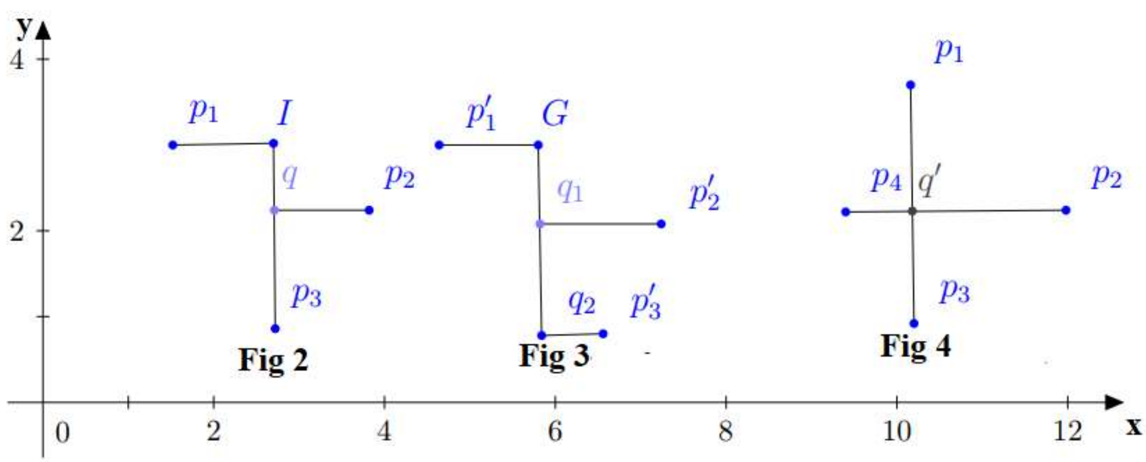
q_i are the additional k vertices in the solution G of S_n, so q_i lie on the edges of G, these straight lines are parallel to the x and y axes. Besides, a straight line connecting at least two points, so that q_i adjacent with 2 p –vertices, we call them p_1 and p_2.

Then, q_i can adjacent 3 p – vertices (see figure 2)

Or q_i can adjacent 2 p- vertices and 1 q – vertices. (see figure 3)

We proof $w(q_i) = 4$

When two pairs of $C(q_i)$ must be collinear and q_i is at the intersection of the straight lines connecting those pairs. (see figure 4)



Condition (ii) $1 \le \mathrm{w}(p_i) \le 4 \;\; 1 \le i \le n$

The first, p_i is one of the given n points and at the vertex of G, so p_i must adjacent with at least one vexter line on the same edge.. Therefore, $\mathrm{w}(p_i) \ge 1$.

The second, p_i can adjacent with at most four vertices when p_i is at the intersection of two edges of G. Therefore, $\mathrm{w}(p_i) \le 4$.

Condition (iii) $0 \le k \le n-2$

In the case, $w(q_i) = 3$ $(1 \le i \le k)$

The number edges of G which have at least one q – vertices is 3k. The other side, the q – vertices form a subtree with (k -1) edges. Since each edge counts twice in the tolal degree of the q- vertices,

$$n \ge 3k - 2(k-1)$$
$$\Leftrightarrow n \ge k+2$$
$$\Leftrightarrow k \le n-2$$

Since $d_{S_n} \geq \frac{1}{2}P(R)$, in the example shown in figure 5, $d_{S_n} \geq \frac{1}{2}P(R)$, we have found a minimum tree with k = 0. Then the solution G of S_n hasn't the additional vertices.

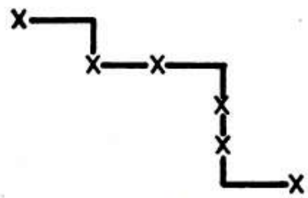


Fig 5

Theorem 3.3. If q is any q – vertex of G with degree three, then q can be the only vertex of G inside the enclosing retangle R of C(q).

4. CONCLUSIONS

Nowadays, the Steiner problem with rectangular distances is widely used in life such as construction, IC technology. To solve this problem the most important thing is to identify the additional q-vertices, also known as Steiner points. In this paper, exact solution are constructed for n =3 and we have given several necessary conditions on the solution to S_n. This is an important basis for solving problems with n >4.

5. REFERENCES

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