OSCILLATION OF SECOND-ORDER HALF-LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT

In this paper, we establish some Oscillation for the second-order half-linear dynamic equation. Our results not only unify the oscillation of half-linear differential and half-linear difference equations but can be applied on different types of time scales and improve some well-known results in the difference equation.

Keyword:- *Oscillation, second-order half-linear dynamic equations, time scale, right-dense, left-dense, right-scattered, left-scattered.*

1.1 INTRODUCTION:

In this paper, we are concerned with oscillation of second-order half-linear dynamic equation

$$(r(u)(y^{\Delta}(u))^{\gamma})^{\Delta} + s(u)\,y^{\gamma}(u) = 0,\ \ t \in [a,b] \qquad \text{…..(1.1.1)}$$

on time scales, where (H) r, s are positive, real-valued rd-continuous functions, and $\gamma > 1$ is an odd positive integer. We shall also consider the two cases

$$\int_{u_0}^{\infty} \left(\frac{1}{r(u)}\right)^{1/\gamma} \Delta u = \infty \qquad \text{…..(1.1.2)}$$

and $$\int_{u_0}^{\infty}\left(\frac{1}{r(u)}\right)^{1/\gamma}\Delta u<\infty \quad \text{.....(1.1.3)}$$

By a solution of (1.1.1), we mean a nontrivial real-valued function $y(u)\in C_{rd}^{1}[u_y,\infty)$, $u_y\geq u_0\geq a$, which has the property $r(u)\left(y^{\Delta}(u)\right)^{\gamma}\in C_{rd}^{1}[u_y,\infty)$ and satisfying equation (1.1.1) for $u\geq u_y$. The Riccati transformation technique, a simple consequence of Keller's chain rule, and the inequality

$$A^{\lambda}-\lambda AB^{\lambda-1}+(\lambda-1)B^{\lambda}\geq 0,\quad \lambda>1, \quad \text{......(1.1.4)}$$

where A and B are nonnegative constants.

1.2 SOME PRELIMINARIES ON TIME SCALES:

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\left.\begin{aligned}\sigma(u)&:=\inf\{t\in\mathbb{T}: t>u\},\\ \rho(u)&:=\sup\{t\in\mathbb{T}, t>u\}.\end{aligned}\right\} \quad \text{.....(1.2.1)}$$

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(u):=\sigma(u)-u$.

For a function $f:\mathbb{T}\to\mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(u)=\frac{f(\sigma(u))-f(u)}{\sigma(u)-u}, \quad \text{.....(1.2.2)}$$

if f is continuous at u and u is right-scattered. If u is not right-scattered then the derivatives is defined by provided this limit exists.

$$f^{\Delta}(u)=\lim_{t\to u}\frac{f(u)-f(t)}{u-t}, \quad \text{.....(1.2.3)}$$

A function $f:[a,b]\to\mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and f is said to be differentialbe if its derivative exists. A useful formula is

$$f(\sigma(u))=f(u)+\mu(u)f^{\Delta}(u). \quad \text{.....(1.2.4)}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient (f/g) (where $gg^{\sigma} \neq 0$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\Delta}, \quad \text{.....(1.2.5)}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \quad \text{.....(1.2.6)}$$

For $a, b \in \mathbb{T}$, and a differentiable equation f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(u)\Delta u = f(b) - f(a). \quad \text{.....(1.2.7)}$$

An integration by parts formula

$$\int_a^b f(u)\, g^{\Delta}(u)\Delta u = [f(u)\, g(u)]_a^b - \int_a^b f^{\Delta}(u)\, g(\sigma(u))\Delta u \quad \text{.....(1.2.8)}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(u)\Delta u = \lim_{b\to\infty} \int_a^b f(u)\,\Delta u.$$

In case $\mathbb{T} = \mathbb{R}$, we have $\sigma(u) = \rho(u) = u$, $\mu(u) = 0$,

$$f^{\Delta} = f' \quad \text{and} \quad \int_a^b f(u)\,\Delta u = \int_a^b f(u)\,du$$

and in case $\mathbb{T} = \mathbb{Z}$, we have $\sigma(u) = u + 1$, $\mu(u) = 1$,

$$f^{\Delta} = \Delta f \quad \text{and} \quad \int_a^b f(u)\,\Delta u = \sum_{u=a}^{b-1} f(u),$$

in the case $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(u) = u + h$, $\mu(u) = h$,

$$f^{\Delta} = \Delta_h f = \frac{f(u+h) - f(u)}{h} \quad \text{and} \quad \int_a^b f(u)\,\Delta u = \sum_{i=a/h}^{b/h-1} f(i);$$

and in the case $\mathbb{T} = q^{\mathbb{N}} = \{u : u = s^k, k \in \mathbb{N}, s > 1\}$, we have $\sigma(u) = su$, $\mu(u) = (s-1)u$,

$$y_s^{\Delta}(u) = \frac{y(su) - y(u)}{(s-1)u} \quad \text{and} \quad \int_a^{\infty} f(u)\Delta u = \sum_{k=0}^{\infty} \mu(s^k) f(s^k).$$

2 MAIN RESULTS:

We suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. Also, we will use the formula

$$(y^{\gamma}(u))^{\Delta} = \gamma \int_0^1 [hy^{\sigma} + (1-h)y]^{\gamma-1} \mathrm{d}h y^{\Delta}(u), \qquad \text{.….(1.3.1)}$$

Theorem 1.4:

Assume that (H) and (1.1.2) hold. Furthermore, assume that there exists a positive Δ-differentiable function $\delta(u)$ such that

$$\limsup_{u\to\infty} \int_u^a \left[\delta(t)s(t) - \frac{r(t)\,(\delta^{\Delta}(t))_+^{\delta+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}\right]\Delta t = \infty \qquad ..$$

where $(\delta^{\Delta}(u))_+ = \max\{0, (\delta^{\Delta}(u))\}$. Then every solution of equation (1.1.1) is oscillatory on $[a, \infty)$.

Proof:

Suppose to the contrary that $y(u)$ is a no oscillatory solution of (1.1.1).

Without loss of generality, we may assume that $y(u)$ is an eventually positive solution of (1.1.1) such that $y(u) > 0$ for all $u \geq u_0 > a$. We shall consider only this case, since the substitution $z(u) = -y(u)$ transforms equation (1.1.1) into an equation of the same form (1.1.1), we have $(r(u)(y^{\Delta}(u))^{\gamma})^{\Delta} = -s(u)y^{\gamma}(u) < 0$, …..(1.4.2)

for all $u \geq u_0$, and so $\{r(u)(y^{\Delta}(u))^{\gamma}\}$ is an eventually decreasing function.

We first show that $\{r(u)(y^{\Delta}(u))^{\gamma}\}$ is eventually nonnegative. since $s(u)$ is a positive function, the decreasing function $r(u)(y^{\Delta}(u))^{\gamma}$ is either eventually positive or eventually negative. Suppose there exists an integer $u_1 \geq u_0$ such that $r(u_1)(y^{\Delta}(u_1))^{\gamma} = c < 0$.

By equation (1.4.2) we have $r(u)(y^{\Delta}(u))^{\gamma} < r(u_1)(y^{\Delta}(u_1))^{\gamma} = c$ for $u \geq u_1$, hence

$$y^{\Delta}(u) \leq c^{1/\gamma}\left(\frac{1}{r(u)}\right)^{1/\gamma},$$

which implies by (1.1.2) that

$$y(u) \leq y(u_1) + c^{1/\gamma} \int_{u_1}^{u} \left(\frac{1}{r(t)}\right)^{1/\gamma} \Delta t \to -\infty \text{ as } u \to \infty, \quad \ldots(1.4.3)$$

which contradicts the fact that $y(u) > 0$ for all $u \geq u_0$. Hence $r(u)(y^{\Delta}(u))^{\gamma}$ is eventually nonnegative.Therefore, we see that there is some u_0 such that $y(u) > 0,\ y^{\Delta}(u) > 0,\ (r(u)(y^{\Delta}(u))^{\gamma})^{\Delta} < 0,$ $u \geq u_0$. …..(1.4.4)

Define the function $w(u)$ by

$$w(u) = \delta(u)\frac{r(u)(y^{\Delta}(u))^{\gamma}}{y^{\gamma}(u)}, \quad u \geq u_0. \qquad \ldots(1.4.5)$$

Then $w(u) > 0$, and using (1.2.5) and (1.2.6) we obtain

$$w^{\Delta}(u) = \frac{\delta(u)}{y^{\gamma}(u)}(r(u)(y^{\Delta}(u))^{\gamma})^{\Delta} + r(y^{\Delta})^{\gamma})^{\sigma}\left[\frac{y^{\gamma}(u)\delta^{\Delta}(u) - \delta(u)(y^{\gamma}(u))^{\Delta}}{y^{\gamma}(u)y^{\gamma}(\sigma(u))}\right] \quad \ldots(1.4.6)$$

From of (1.1.1) and (1.4.6), we get

$$w^{\Delta}(u) = -\delta(u)r(u) + \frac{\delta^{\Delta}(u)}{\delta^{\sigma}}w^{\sigma} - \frac{\delta(u)(r(y^{\Delta})^{\gamma})^{\sigma}(y^{\gamma}(u))^{\Delta}}{y^{\gamma}(u)y^{\gamma}(\sigma(u))} \qquad \ldots(1.4.7)$$

Using (1.4.3) we have $y^{\sigma} \geq y(u)$, and then from the chain rule (1.3.1) we obtain

$$(y^{\gamma}(u))^{\Delta} = \gamma\int_0^1 [hy^{\sigma} + (1-h)y]^{\gamma-1}y^{\Delta}(u)\,dh$$

$$\geq \gamma\int_0^1 [hy + (1-h)y]^{\gamma-1}y^{\Delta}(u)\,dh$$

$$= \gamma(y(u))^{\gamma-1}y^{\Delta}(u). \qquad \ldots(1.4.8)$$

It follows that from (1.4.7) and (1.4.8) that

$$w^{\Delta}(u) \leq -\delta(u)r(u) + \frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}}w^{\sigma} - \frac{\delta(u)(r(y^{\Delta})^{\gamma})^{\sigma}\gamma(y(u))^{\gamma-1}y^{\Delta}(u)}{y^{\gamma}(u)y^{\gamma}(\sigma(u))}$$

$$= -\delta(u)r(u) + \frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}} w^{\sigma} - \frac{\gamma\delta(u)(r(y^{\Delta})^{\gamma})^{\sigma} y^{\Delta}(u)}{y(u)y^{\gamma}(\sigma(u))}$$

$$\leq -\delta(u)r(u) + \frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}} w^{\sigma} - \frac{\gamma\delta(u)(r(y^{\Delta})^{\gamma})^{\sigma} y^{\Delta}(u)}{y^{\gamma+1}(\sigma(u))} \quad \ldots..(1.4.9)$$

From (1.4.4) since $(r(u)(y^{\Delta}(u))^{\gamma})^{\Delta} < 0$ we have

$$y^{\Delta}(u) > \frac{(r^{\sigma})^{1/\gamma}}{r^{1/\gamma}} (x^{\Delta})^{\sigma}. \quad \ldots..(1.4.10)$$

Substituting (1.4.10) in (1.4.9) we find that

$$w^{\Delta}(u) < -\delta(u)r(u) + \frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}} w^{\sigma} - \frac{\gamma\delta(u)(r^{\sigma})^{(\gamma+1)/\gamma}(y^{\Delta})^{\gamma+1}(\sigma(u))}{r^{1/\gamma}\, y^{\gamma+1}(\sigma(u))}$$

$$= -\delta(u)s(u) + \frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}} w^{\sigma} - \frac{\gamma\delta(u)}{(\delta^{\delta})^{\lambda} r^{\lambda-1}(u)} (w^{\sigma})^{\lambda}, \quad \ldots.(1.4.11)$$

where $\lambda = (\gamma+1)/\gamma$.

Let

$$A = \left[\frac{\gamma\delta(u)}{(\delta^{\delta})^{\lambda} r^{\lambda-1}(u)}\right]^{1/\lambda} w^{\sigma}$$

and

$$B = \left[\frac{(\delta^{\Delta}(u))_+}{\lambda\delta^{\sigma}} \left(\frac{\gamma\delta(u)}{(\delta^{\delta})^{\lambda} r^{\lambda-1}(u)}\right)^{-1/\lambda}\right]^{1/(\lambda-1)}$$

Using inequality (1.1.4), we have

$$\frac{(\delta^{\Delta}(u))_+}{\lambda\delta^{\sigma}} w^{\sigma} - \frac{\gamma\delta(u)}{(\delta^{\delta})^{\lambda} r^{\lambda-1}(u)} (w^{\sigma})^{\lambda}$$

$$\leq (\lambda-1)\lambda^{\lambda/(\lambda-1)} \left(\frac{(\delta^{\Delta}(u))_+}{\delta^{\sigma}}\right)^{\lambda/(\lambda-1)} \left(\frac{\gamma\delta(u)}{(\delta^{\delta})^{\lambda} r^{\lambda-1}(u)}\right)^{-1/(\lambda-1)}$$

$$= C\frac{r(u)(\delta^{\Delta}(u))_{+}^{\lambda/(\lambda-1)}}{\delta^{1/(\lambda-1)}(u)}$$

$$= C\frac{r(u)(\delta^{\Delta}(u))_{+}^{\gamma+1}}{\delta^{\gamma}(u)}, \qquad \ldots\ldots(1.4.12)$$

where $C = (\lambda-1)\lambda^{\lambda/(\lambda-1)}\gamma^{-1/(\lambda-1)} = 1/(\gamma+1)^{\gamma+1}$.

Thus, from (1.4.11) and (1.4.12) we obtain

$$w^{\Delta}(u) < -\left[\delta(u)r(u) - \frac{r(u)(\delta^{\Delta}(u))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(u)}\right]. \qquad \ldots\ldots(1.4.13)$$

Integrating (1.4.13) from u_0 to u, we obtain

$$-w(u_0) < w(u) - w(u_0) <$$

$$-\int_{u_0}^{u}\left[\delta(t)r(t) - \frac{r(t)(\delta^{\Delta}(t))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}\right]\Delta t \qquad \ldots\ldots(1.4.14)$$

We have

$$\int_{u_0}^{u}\left[\delta(t)r(t) - \frac{r(t)(\delta^{\Delta}(t))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}\right]\Delta t < w(u_0),$$

for all large u. This is contrary to

$$\limsup_{u\to\infty}\int_{u}^{a}\left[\delta(t)r(t) - \frac{r(t)(\delta^{\Delta}(t))_{+}^{\delta+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}\right]\Delta t = \infty$$

Hence the proof.

Theorem 1.5:

Assume that (H) and (1.1.3) hold. Let $\delta(u)$ be as defined in Theorem(1.4) such that (1.4.1) holds. If

$$\int_a^\infty \left[\frac{1}{r(u)}\int_a^u s(t)\,\Delta t\right]^{1/\gamma} \Delta u = \infty \qquad \ldots..(1.5.1)$$

Then every solution of equation (1.1.1) is oscillatory or converges to zero.

Proof:

We assume that equation (1.1.1) has a nonoscillatory solution such that $y(u) > 0$, for $u \geq u_0 > a$.

(we shall consider only this case, since the substitution

$z(u) = -y(u)$ transforms equation(1.1.1) into an equation of the same form.)

From the proof of theorem (1.4)

we see that there exist two possible cases for the sign of $y^\Delta(u)$.

The proof when $y^\Delta(u)$ is eventually positive is similar to that of the proof of Theorem (1.4)

Next, suppose that $y^\Delta(u) < 0$ for $u \geq u_1 \geq u_0$.

Then $y(u)$ is decreasing and $\lim_{u\to\infty} y(u) = b \geq 0$ exists.

We assert that $b = 0$. If not, then $y(u) > b > 0$ for $u \geq u_2 > u_1$.

Define the function

$$v(u) = r(u)\,(y^\Delta(u))^\gamma,$$

Then from equation (1.1.1) for $u \geq u_2$, we obtain

$$v^\Delta(u) = -s(u)\,y^\gamma(u)$$

$$\leq -b^\gamma s(u).$$

Hence, for $u \geq u_2$ we have

$$v(u) \leq v(u_2) - b^\gamma \int_{u_2}^u s(t)\Delta t$$

$$< -b^\gamma \int_{u_2}^u s(t)\Delta t.$$

Since $v(u_2) = r(u_2)(y^{\Delta}(u_2))^{\gamma} < 0$,

Integrating the last inequality from u_2 to u, we have

$$\int_{u_2}^{u} y^{\Delta}(t)\Delta t \leq -b \int_{u_2}^{u} \left[\frac{1}{r(t)} \int_{u_2}^{t} s(\tau)\Delta\tau\right]^{1/\gamma} \Delta t$$

By condition (1.5.1), we get $y(u) \to -\infty$ as $u \to \infty$, and this is a contradiction to the fact that $y(u) > 0$ for $u \geq u_0$.

Thus $b = 0$ and $y(u) \to 0$ as $u \to \infty$.

Hence the proof.

CONCLUSION

In this paper, by using the chain rule and the Riccati transformation technique, we have established some new oscillation of second-order half-linear dynamic equations on time scales. Our results not unify the oscillation of differential and difference equations but also improve the results of second-order half-linear difference equations.

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