Review of Some Nonlinear Partial Differential Equations

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Abstract

In this Paper, we discuss some nonlinear partial differential equations (NPDE) like Burgers equations, Boussinesq equation, Camassa-Holm equations, Fishers equations, Clairauts equations etc. These NPDE are applicable in various fields.Some equations among these are strongly non-linear which will be solve by using different direct and indirect methods.

Keywords: Nonlinear Partial differential equations, Variational iteration method, Lagrange multiplier, Correction functional.

Introduction

In real world, nonlinear phenomena that seem in many applications like fluid dynamics, plasma physics, geophysics, solid state physics, quantum mechanics, electricity and magnetism, kinematic theory of gases, mathematical economics.

In this paper, some nonlinear partial differential equations are reviewed.The NPDE like The Burgers Equation, Benjamin-Bona-Mahony Equation, The Boussinesq Equation, The Camassa-Holm (CH) equation, The Integro Differential Equation, The Riccati Differential Equation, The Bratu-Type Equation, The Kortweg-de Vries (KdV) Equation, The Fisher Equation, The Schrodinger Equation, The Zakharor-Kuznetsov (ZK) Equation, The Sine-Gordon (SG) Equation, The Gardner Equation, The Zhiber-Shabat (ZS) Equation, The Klein-Gordon (KG) Equation, The Hirota-Satsuma (HS) Equation, The Kawahara Equation, The Kuramoto-Sivashinsky (KS) Equation, The Ginzburg-Landau (GL) Equation, The Kadomtsov-Petviashrili (KP) Equation, The K(n, n) Equation, The Medium Equal Width (MEW) Equation are discussed in second section.The conclusion is given in last section.

Some Nonlinear Partial Differential Equations

1. The Burgers Equation:

Harry Bateman introduced this equation in 1915, after that Johannes Martinus Burgers studied it in 1948. Therefore, Burgers equation is also known as Bateman-Burgers equation.

It occurs in gas dynamics, traffic flow, non linear acoustics and fluid mechanics. The general form of Burgers equation is given by

$$w_t + ww_x = uw_{xx} \tag{1.1}$$

Where $w = w(x,t)$ is any field and u is diffusion coefficient.

Equation (1.1) is also called as viscous Burgers equation.

If $u = 0$ then equation (1.1) becomes

$$w_t + ww_x = 0 \tag{1.2}$$

It is known as Inviscid Burgers equation which is advective form of the Burgers equation.

Conservative form of Burgers equation is given by

$$w_t + \frac{1}{2} w_{xx} = 0$$

Burgers equation in other Forms :

1) **Generalized Burgers equation:**
The equation of the form

$$w_t + f(w) w_x = u\, w_{xx} \tag{1.3}$$

where f(w)is function of w , is generalized Burgers equation.

2) **Stochastic Burgers equation :-**
The stochastic Burgers equation is given by

$$w_t + w . w_x = u . w_{xx} - \lambda\, \xi_x \tag{1.4}$$

where $\xi(x, t)$ is space- time noise term.

Equation (1.4) is the dimensional Kardar-Parisi-Zhang equation (KPZ).

2. Benjamin-Bona-Mahony Equation:

The partial differential equation of the type

$$w_t + w_x + w . w_x - w_{xxt} = 0$$

is known as Benjamin-Bona-mahony (BBM) equation.

A generalized form is given as

$$w_t - \nabla^2 w_t + \operatorname{div} (\phi\, (w) = 0$$

The BBM equation was introduced by peregrine in 1966.This equation is improvement of Korteweg-de-Vries (KdV) equation for showing elongated surface gravity waves of minor amplitude.This equation is also known as Regularized long-wave equation.

3. The Boussinesq Equation:-

For water waves, Boussinesq approximation is an approximation valid for weakly nonlinear and fairly long waves in fluid dynamics. Joseph Valentin Boussinesq (1842-1929) first derived these approximation to the observation by john scott Russell of solitary wave.

According to 1872 papers of Boussinesq, derivation of this equation is given as follows:

In the XZ-Plane for water waves on an in compressible fluid and irrotational flow, the boundary condition at the free surface elevation z $= \xi(x, t)$ are

$$\frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} - v = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + v^2) + g\xi = 0$$

Where, $u = \frac{\partial \phi}{\partial x}$ = horizontal flow velocity component

$v = \frac{\partial \phi}{\partial z}$ = vertical flow velocity component

g = acceleration due to gravity

In above boundary conditions, if we applied Boussinesq approximations for velocity potential ϕ, only linear and quadratic terms with respect to ξ and u_b are taken where $u_b = \frac{\partial \phi_b}{\partial x}$ = horizontal velocity.

The cubic and higher order terms supposed to be negligible. Hence we get partial differential equation as -

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}[(h + \xi)u_b] = \frac{1}{6}h^3\frac{\partial^3 u_b}{\partial x^3}$$
$$\frac{\partial u_b}{\partial t} + u_b\frac{\partial u_b}{\partial x} + g\frac{\partial \xi}{\partial x} = \frac{1}{2}h^2\frac{\partial^3 u_b}{\partial t\,\partial x^2}$$

From the above two equation we get partial differential equation,

$$\frac{\partial^2 \xi}{\partial t^2} - gh\frac{\partial^2 \xi}{\partial x^2} - gh\frac{\partial^2}{\partial x^2}\left(\frac{3}{2}\frac{\xi^2}{h} + \frac{1}{3}h^2\frac{\partial^2 \xi}{\partial x^2}\right) = 0 \tag{1.5}$$

If quantities are dimensionless and if we take,

$\alpha = \frac{1}{2}\frac{\xi}{h}$=dimensionless surface elevation.

$\beta = t\sqrt{\frac{3g}{h}}$= dimensionless time.

$\gamma = \sqrt{3}\frac{x}{h}$= dimensionless horizontal position.

then from equation (1.5) we get,

$$\frac{\partial^2 \alpha}{\partial \beta^2} - \frac{\partial^2 \alpha}{\partial \gamma^2} - \frac{\partial^2}{\partial \gamma^2}\left(3\alpha^2 + \frac{\partial^2 \alpha}{\partial \gamma^2}\right) = 0 \tag{1.6}$$

Recently, equation (1.6) is written as,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left(3u^2 \pm \frac{\partial^2 u}{\partial x^2}\right) = 0$$

Hence, $u_{tt} - u_{xx} - (3u^2 \pm u_{xx})_{xx} = 0$

$$u_{tt} - u_{xx} \pm u_{xxxx} = 3u^2 \tag{1.7}$$

equation (1.7) with positive sign is good boussinesq or well posed boussinesq equation and with negative sign is bad or ill-posed bounssinesq equation.

4. The Camassa-Holm (CH) equation:

In 1993, Roberto camassa and Darryl Holm have found nonlinear wave equation

$$w_t + 2k\, w_x - w_{xxt} + 3ww_x = 2\, w_x w_{xx} + w\, w_{xxx} \tag{1.8}$$

where k is positive, and solutions are solitary waves.

If $k = 0$ then equation (1.8) has peakons solutions with sharp peak. In the wave slope discontinuity occurs at the peak.

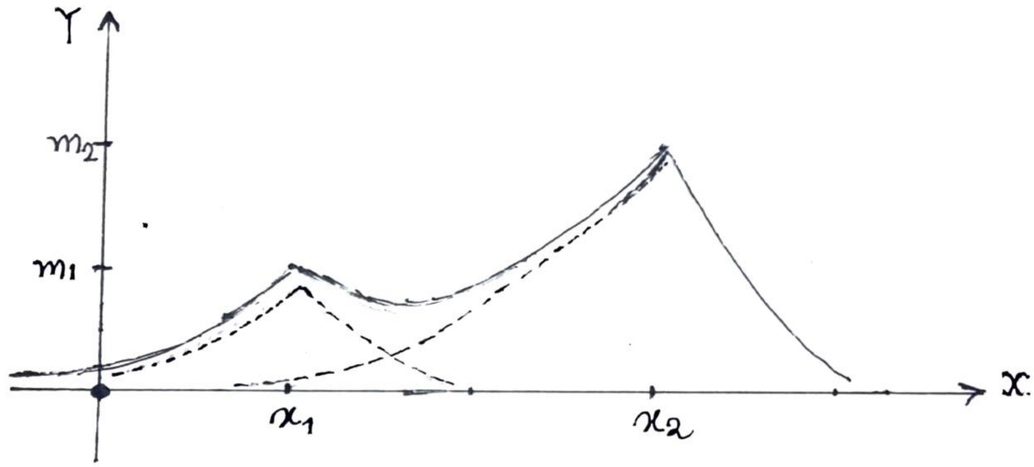

Figure:1. Interaction of two peakons.

In figure1.2, interaction of two peakons is sharp solitary wave solution to the CH-equation. Solid curve is formed by addition of two dotted curves.

$$\mathrm{w} = \mathrm{m}_1 e^{-|x-x_1|} + \mathrm{m}_2 e^{-|x-x_2|}$$

where, x_1, $x_2$are positionof peakons and m_1, m_2 are peakons amplitudes. Camassa-Holm equation occurs in fluid dynamics.

5. The Integro Differential Equation:

In many phenomena such as fluid dynamics, chemical kinetics, dropwise condensation, wind ripple in desert region, process of glass forming and nano-hydrodynamics integro-differential equations occurs.These equation are difficult to solve analytically , hence approximate solution is required.

Following are integro-differential equation:

1) The volterra integro-differential (VID)equation is given as

$$\frac{\partial w}{\partial x} = g(x) + \int_0^x \phi\ (\mathrm{x,t,w(t),w'(t)})\ \mathrm{dt}$$

Where w(x) is function which we have to determine. g(x) is given function.

2) Fredholm integro-differential (FID) equationis given as

$$\frac{\partial w}{\partial x} = g(x) + \int_c^d \phi\ (\mathrm{x,t,w(t),w'(t)})\ \mathrm{dt}$$

Generally, non-linear VID equation is given by

$$\frac{\partial^n w}{\partial x^n} = g(x) + \lambda \int_0^x \phi\ (\mathrm{x,t})\ \mathrm{G(w,w')dt}\ ;\ \mathrm{c}\ \leq x \leq d$$

and general FID equation is given by

$$\frac{\partial^n w}{\partial x^n} = g_i(x) + \sum_{\mathrm{i=1}}^{\mathrm{n}} \lambda i\,[\phi_i\ (\mathrm{x,t})\ \mathrm{G(w(t))}]\mathrm{dt}$$
$$\mathrm{w^k(0) = q_k}\ ;\ 0\ \leq k \leq (n-1)$$

Where $\mathrm{q_k}$is constants. G(w(t)) is nonlinear. $w\ (x)$, $g_i(x)$ are real functions and we have to determined G, g_i, ϕ_i which are continuous functions.

Volterra-fredholm Integral (VFI) equation is given by

$$\phi(x) - \lambda_1 \int_0^x \phi_1(x,w)\, M_1(\phi\,(w))dw - \lambda_2 \int_c^d \phi_2(x,w)\, M_2(\phi\,(w))dw = g(x)$$

Here $g(x), \phi_1$ and ϕ_2 are defined on [0, 1].

6. The Riccati Differential Equation:

The name Riccati Differential equation (RDE) [3] is given due to Italian nobleman count Jacopo Franscesco Riccati. This equation is nonlinear differential equation given as

$$\frac{dw}{dt} = a(t)y + b(t)\,y^2 + c(t)\,;\; w(0) = \beta$$

Where$a(t), b(t), c(t)$ are scalar functions, where as β is constant. The RDE is used in optimal control dynamic games and also in diffusion problems. In stead of all these applications RDE is also applicable in synthesis of network, stabilization of robust and stochastic realization theory. Recently there is a lot of use of RDE in financial Mathematics. Hence it is very important in Engineering and applied science. The schrondinger equation of one dimension is nearly closed to Riccati Differential equation..

7. The Bratu-Type Equation:

In one dimensional, the boundary value problem which is Bratu-type [2, 6] is given by

$$\frac{\partial^2 w}{\partial t^2} + \beta\, e^w = 0\,;\; 0 < t < 1, \beta > 0$$

with $w(0) = w(1) = 0$

The Bratu type initial value problem as follows

$$\frac{\partial^2 w}{\partial t^2} - 2\, e^w = 0$$

with $w(0) = w'(0) = 0$

This equation extensively studied because of its physical also mathematical properties. In thermal combustion theory these equation arises for simplification of solid fuel ignition model. Also these equation arises in chemical reaction theory and to the expansion of universe nanotechnology; radioactive heat transfer. Bratu type equation is applicable in numerical slab to a model of combustion problem.

8. The Kortweg-de Vries (KdV) Equation:

The kortweg-de vries (KdV) equation is in the form,

$$w_t + \alpha\, ww_x + \beta\, w_{xxx} = 0$$

whereα and $\beta$are constant.

The generalkortweg-de vries (gKdV) equation is given by

$$w_t + \gamma w^a w_x + \delta w_{xxx} = 0$$

where γ, δ are constants and a is positive integers.

If we take $a = 2$ then gkdv is known as modified kortweg-de vries (mKdV) equation. The KdV equation in generalized form is given as

$$w_t + (a+1)(a+2)w^a w_x + w_{xxx} = 0, a = 1,2,3, \ldots$$

$w(x, 0) = f(x)$

If we substitute $w = w_x$ in KdV equation we get

$$w_t + \alpha . w_x{}^2 + \beta . w_{xxx} = 0$$

It is known as potential KdV equation.In the study of shallow water waves KdV equation occurs.

9. The Fisher Equation:

The fisher equation [4] is expressed as

$$w_t - w(1 - w) - w_{xx} = 0$$

The nonlinear problems of evolution of a population in one dimensional habitual, nuclear reaction includes neutron population are present in population dynamics and chemical kinetics. In such cases the fisher equation occurs. Also, the process of interaction between diffusion and reaction is described by fisher equation.

10. The Schrodinger Equation:

The Schrodinger equation is in both linear and nonlinear form.

ConsiderLinear Schrodinger equation,

$$w_t = iw_{xx};\ w(x,0) = g(x), t > 0$$

where i is imaginary unit and $g(x)$ is square integrable continuous function.

Also, nonlinear Schrodinger (NSE) equation is given by

$$w_{xx} + iw_t + \alpha\, w|w|^2 = 0$$

where $w(x, t)$ is complex and α is constant.

The NSE will be handleby Variational interation method (VIM) and Adomian Decomposition method (ADM).

Commonly, NSE are given as

$$w_{xx} + iw_t \pm 2\, w|w|^2 = 0$$

Depending on value of αother forms of NS equation are used.

11. The Zakharor-Kuznetsov (ZK) Equation :

The zakharor-kuznetsov (ZK) Equation is expressed as

$$w_t + \alpha\, ww_x + \beta(w_{xx} + w_{yy} + w_{zz})_x = 0$$

the generalization of KdV equation is nothing but ZK equation. In two dimension, the strongly magnetized lossless plasma which includes poorly nonlinear ion-acoustic waves are given by firstly by ZK equation. Also in the presence of uniform magnetic field these waves comprising hot isothermal electrons and cold ions.

We cannot find soliton solution of ZK equation due to these equation is non integrable.Alsoother solution of these equation are inelastic.

12. The Sine-Gordon (SG) Equation:

The Sine-Gardon equation (SGE) [7] is expressed in standard form as

$$w_{tt} - \alpha^2 w_{xx} + \beta \sin w = 0$$

$w(x,0) = p(x)$ and $w_1(x,0) = q(x)$.

where α, β are real numbers. Firstly, in differential geometry SGE occurs.Many physical concept like fluid motions and propagation of magnetic flux contains SGE. hence, recently a lot of research work, is going on these equations.

In nonlinear physics, SGE plays important role. The SGE is not wave problems in first occurrence but surface with Gaussian curvature ($\kappa = -1$) in the differential geometry these equations are integrable totally.

13. The Gardner Equation:

The Gardner equation in its standard form is given by

$$w_t + 2pww_x - 3qw^2w_x + w_{xx} = 0\,;\ \ p, q > 0$$

The KdV and mKdV equation together we get the Gardner equation. Initially these equations are derived in a two layer fluid with density jump at interface for long internal waves in asymptotic theory. These equations are mostly applicable in fluid physics, quantum field theory and plasma physics. Gardner equation is applicable to nonlinear concepts in variety. This equation is interesting due to cubic nonlinearities, quadratic nonlinearities and among dispersion.

Gardner equation is integrable totally as like KdV and mKdV equations with inverse scattering transform and lax pair. For completely integrable Gardner equation, there are two equations.

$$w_t + 6ww_x \pm 6w^2w_x + w_{xxx} = 0$$

signs of cubic nonlinear term gives positive and negative Gardner equation depending on positive and negative signs respectively.

14. The Zhiber-Shabat (ZS) Equation:

The Zhiber-Shabat (ZS) equation is expressed as

$$w_{xt} + ae^{w} + be^{-w} + ce^{-2w} = 0$$

where a, b and c are real numbers.

If $b = 0$ then we get,

$$w_{xt} + ae^{w} + ce^{-2w} = 0$$

It is known as Dodd-Bullough-Mikhailor (DBM) Equation.

If $a = 0, b = -1$ and $c = 1$ then we get,

$$w_{xt} - e^{-w} + e^{-2w} = 0$$

This equation is Tzitzeica-Dodd-Bullougs (TDB) equation.

If $c = 0$ then we get

$$w_{xt} + ae^{w} + be^{-w} = 0$$

It is nothing but sinh-Gordon equation.

In various application like chemical kinetics, plasma physics, quantum field theory, non-linear optics, solid state physics the above all ZK equations occurs.

15. The Klein-Gordon (KG) Equation:

The Klein-Gordon (KG) equations [5] are in both linear and non-linear form.

a) Linear KG Equation is expressed as

$$w_{tt} - w_{xx} + \alpha w = r(x,t)$$

subject to IC: $w(x,0) = p(x), w_t(x,0) = q(x)$
where α is real number and source term is $r(x,t)$.
if $\alpha = 0$,then these equation isnon-homogeneous wave equation from relativistic energy formula. In quantum mechanics we get its derivation which plays an important role.

b) The non linear KG equation is expressed in standard form as

$$w_{tt} - w_{xx} + \alpha w + G(w) = r(x,t)$$

with IC: $w(x,0) = p(x), w_t(x,0) = q(x)$
where α is real constants and $r(x,t)$ is source term,$G(w)$ is nonlinear function of $w = w(x,t)$
By some numerical techniques, investigation of these equation will be done.

16. The Hirota-Satsuma (HS) Equation:

The Hirota-Satsuma equation is given by

$$w_t = \frac{1}{2} w_{xxx} + 3ww_x - 6yy_x$$

$$y_t = -y_{xxx} - 3wy_x$$

we get KdV equation, if we put $y = 0$. Hence coupled KdV equation is proposed by Hirota and Satsuma, gives description of interaction of two waves which is long with relations of different dispersion.

17. The Kawahara Equation :

The Kawahara equation in its standard form is given by

$$w_t + 6ww_x + w_{xxx} - w_{xxxxx} = 0$$

Actually, this equation is fifth order KdV equation. Kawahara after observation conclude that these equations have monotonic solitary and oscillatory solutions. These equations also describe capillary- gravity water waves and model for plasma waves.

For model of magnetic acoustic waves, the dispersive term w_{xxxxx} is necessary.It also contains term ww_xwhich also occurs in Burgers, KdV equations. In the theory of magneto-acoustic waves in cold collision free plasma these equations presents. Also it occurs in shallow water waves theory with surface tension.

Modified Kawahara equation:

Modified Kawahara equation is given by

$$w_t + 6w^2 w_x + w_{xxx} - w_{xxxxx} = 0$$

It is also known as singularity perturbed KdV equation.

Here, w_{xxx}, w_{xxxxx} are two dispersive terms and $(w^3)_x$ is non-linear term.

18. The Kuramoto-Sivashinsky (KS) Equation :

The Kuramoto-Sivashinsky(KS) [1] equation is expressed in the form

$$w_t + \alpha ww_x + \beta w_{2x} + \gamma w_{4x} = 0$$

In the context of plasma instabilities, phase turbulence in reaction-diffusion systems and flame front propagation this equation was derived. In one dimension this equation is prototypical example of spatiotemporal chaos.

In a homogeneous system spatially uniform oscillating chemical reaction, motion of a fluid going down a vertical wall or fluctuations of flame front. These all description are given by KS equation.

19. The Ginzburg-Landau (GL) Equation:

The Ginzburg-Landau cubic complex equation is given as

$$v_t = (1 + ip)v_{xx} + Qv - (1 + iq)|v|^2 v$$

where p, q and Q are constants. v is complex function from $\mathbb{R}^2$ to $\mathbb{C}$. $x$is real, $t \geq 0$.

The modulations of the field of oscillator is described by v, which is complex field and q , Q are two real control parameters.

In the study of nonlinear optics, reaction-diffusion system, chemical turbulence and hydro-dynamical stability problems GL equation is applicable.

The Generalized nonlinear complex GL equation of $(2n + 1)$ order is expressed as

$$v_t = (1 + ip)v_{xx} + Qv - (1 + iq)|v|^{2n} v$$

where n is natural number.

The generalized nonlinear complex GL equation of $(4n + 1)$ order is given by

$$v_t = (1 + ip)v_{xx} + Qv - (1 + iq)|v|^{2n} v - (1 + ir)|v|^{4n} v$$

whereris constant.

20. The Kadomtsov-Petviashrili (KP) Equation:

The KP equation is given as

$$(w_t - 6ww_x + w_{xxx})_x + 3w_{yy} = 0$$

This is generalization of KdV equation to two space variables formulated by kadomtsov and Petviashrili in 1970.For modeling of shallow water waves with poorly nonlinear restoring force the KP equation is applicable.

A Decomposition method with Adomian polynomials is applicable to solve specific KP equation,

$$w_{xt} - 6w_x^2 - 6ww_{xx} + w_{xxxx} + 3w_{yy} = 0$$

with initial condition

$$w(x, y, 0) = \frac{-8e^{2x+2y}}{(1 + e^{2x+2y})^2}$$

and zero boundary condition.

21. The K (n, n) Equation:

The K (n, n) Equation has the form

$$w_t + \alpha(w^n)_x + (w^n)_{xxx} = 0\, , n > 1$$

There are three variants of K(n, n) equation.

The first variant is discovered by Rosenau, which is given by

$$w_t + \alpha(w^{n+1})_x + [w(w^n)_{xx}]_x = 0\, , \alpha > 0, \qquad n \geq 1$$

This equation is also considered as variant of KdV equation.

The K (n, n) equation in second variant is given as

$$\mathrm{w_t} + \alpha(w^{n+1})_x + [w^n w_{xx}]_\mathrm{x} = 0\,, \alpha > 0, \qquad n \geq 2$$

The third variant of K(n, n) equation is given by

$$\mathrm{w_t} + (\alpha w + \beta\, w^n)_\mathrm{x} + \mathrm{K}(w^n)_{xxx} = 0\,, \qquad \mathrm{n} > 1\,, \mathrm{K} \neq 0$$

If $\mathrm{n} = 1$ and $\beta = 0$ then above equation becomes linear KdV equation.

22. The Medium Equal Width (MEW) Equation:

The medium equal width equation is expressed in the form

$$\mathrm{w_t} + 3w^2 w_x - \alpha w_{xxt} = 0$$

The MEW equation is related to the Regularized long wave (RLW) equation which is given by

$$\mathrm{w_z} + \mathrm{w_t} + \beta\, \mathrm{ww_z} - \eta w_{zzt} = 0$$

where $\beta\,,\eta$ are positive constants.

The MEW equation has pulse like wave solution.Also this is cubic nonlinear wave equation. In the model of nonlinear dispersive waves and many physical phenomeno on this equation is applicable.

Conclusion:

In this article, discussion of nonlinear partial differential equations is done.Some of these nonlinear partial differential equations are strongly nonlinear. There are a lot of application of these NPDE such as Riccati Differential equation isc used in financial Mathematics. Hence it is very important in Engineering and applied science. Bratu type equation is applicable in numerical slab to a model of combustion problem. In the study of shallow water waves KdV equation occurs. The process of interaction between diffusion and reaction is described by fisher equation. In various application like chemical kinetics, plasma physics, quantum field theory, non-linear optics,solid state physics the ZK equations occurs. Hirota and Satsuma, gives description of interaction of two waves which is long with relations of different dispersion. In the theory of magneto-acoustic waves in cold collision free plasma these NPDE presents. Also it occurs in shallow water waves theory with surface tension. The MEW equation has pulse like wave solution. Also this is cubic nonlinear wave equation. In the model of nonlinear dispersive waves and many physical phenomenons this equation is applicable.

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