SECOND ORDER LINEAR DIFFERENTIAL EQUATION-A STUDY

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ABSTRACT; Given real-valued functions a_o, a_1 and b such that $a_o(t)$, $a_1(t)$ and $b(t) \in \mathbf{R}\ \forall\ t \in \mathbf{R}$, the differential equation of the form $y'' + a_1(t)\, y' + a_o(t)\, y = b(t)$ is known as a second-order linear differential equation with variable coefficients. The variable coefficients are $a_o(t)$ and $a_1(t)$. If $b(t) = 0$ then the above equation is called a homogeneous second-order differential equation. In this section give an in depth discussion on the process used to solve homogeneous, linear, second order differential equations, $ay''+by'+cy=0''+=0$. We derive the characteristic polynomial and discuss how the Principle of Superposition is used to get the general solution. In this section we will a look at some of the theory behind the solution to second order differential equations. We define fundamental sets of solutions and discuss how they can be used to get a general solution to a homogeneous second order differential equation. We will also define the Wronskian and show how it can be used to determine if a pair of solutions are a fundamental set of solutions.

KEYWORDS; SECOND ORDER LINEAR, DEFFIRENTIAL EQUATION COEFFICIENT VARAIBLE PRINCIPLE OF SUPERPOSITION

INTRODUCTION

Definition

Given real-valued functions a_o, a_1 and b such that $a_o(t)$, $a_1(t)$ and $b(t) \in \mathbf{R}\ \forall\ t \in \mathbf{R}$, the differential equation of the form

$y'' + a_1(t)\, y' + a_o(t)\, y = b(t)$ is known as a second-order linear differential equation with variable coefficients. The variable coefficients are $a_o(t)$ and $a_1(t)$. If $b(t) = 0$ then the above equation is called a homogeneous second-order differential equation.

Here,

$$y'' = \frac{d^2y}{dx^2} and y' = \frac{dy}{dx}$$

For example, y” + 2y’ + 6 = 0 is a second-order linear differential equation with constant coefficient.

y” + 2t y’ + $\log_e$ t y = e^{3t} is a second-order differential equation with variable coefficients.

In the previous chapter we looked at first order differential equations. In this chapter we will move on to second order differential equations. Just as we did in the last chapter we will look at some special cases of second order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we’ll look at. This will be required in order for us to actually be able to solve them.

Basic Concepts – In this section give an in depth discussion on the process used to solve homogeneous, linear, second order differential equations, ay″+by′+cy=0″+=0. We derive the characteristic polynomial and discuss how the Principle of Superposition is used to get the general solution.

Real Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0′+=0, in which the roots of the characteristic polynomial, ar2+br+c=0^2+=0, are real distinct roots.

Complex Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0″+′+=0, in which the roots of the characteristic polynomial, ar^2+br+c=0^2+=0, are complex roots. We will also derive from the complex roots the standard solution that is typically used in this case that will not involve complex numbers.

Repeated Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0′+=0, in which the roots of the characteristic polynomial, ar^2+br+c=0^2=0, are repeated, *i.e.* double, roots. We will use reduction of order to derive the second solution needed to get a general solution in this case.

Reduction of Order – In this section we will take a brief look at the topic of reduction of order. This will be one of the few times in this chapter that non-constant coefficient differential equation will be looked at.

Fundamental Sets of Solutions – In this section we will a look at some of the theory behind the solution to second order differential equations. We define fundamental sets of solutions and discuss how they can be used to get a general solution to a homogeneous second order differential equation. We will also define the Wronskian and show how it can be used to determine if a pair of solutions are a fundamental set of solutions.

More on the Wronskian – In this section we will examine how the Wronskian, introduced in the previous section, can be used to determine if two functions are linearly independent or linearly dependent. We will also give and an alternate method for finding the Wronskian.

Nonhomogeneous Differential Equations – In this section we will discuss the basics of solving nonhomogeneous differential equations. We define the complimentary and particular solution and give the form of the general solution to a nonhomogeneous differential equation.

Undetermined Coefficients – In this section we introduce the method of undetermined coefficients to find particular solutions to nonhomogeneous differential equation. We work a wide variety of examples illustrating the many guidelines for making the initial guess of the form of the particular solution that is needed for the method.

Variation of Parameters – In this section we introduce the method of variation of parameters to find particular solutions to nonhomogeneous differential equation. We give a detailed examination of the method as well as derive a formula that can be used to find particular solutions.

Mechanical Vibrations – In this section we will examine mechanical vibrations. In particular we will model an object connected to a spring and moving up and down. We also allow for the introduction of a damper to the system and for general external forces to act on the object. Note as well that while we example mechanical vibrations in this section a simple change of notation (and corresponding change in what the quantities represent) can move this into almost any other engineering field

Second order differential equation is a specific type of differential equation that consists of a derivative of a function of order 2 and no other higher-order derivative of the function appears in the equation. It includes terms like y'', d^2y/dx^2, $y''(x)$, etc. which indicates the second order derivative of the function. Generally, we write a second order differential equation as $y'' + p(x)y' + q(x)y = f(x)$, where $p(x)$, $q(x)$, and $f(x)$ are functions of x. We can solve this differential equation using the auxiliary equation and different methods such as the method of undetermined coefficients and variation of parameters.

The differential equation y'' + p(x)y' + q(x)y = 0 is called a second order differential equation with constant coefficients if the functions p(x) and q(x) are constants and it is called a second-order differential equation with variable coefficients if p(x) and q(x) are not constants. In this article, we will understand such differential equations in detail and their different types. We will also learn different methods to solve second order differential equations with constant coefficients using the various methods with the help of solved examples and finding the auxiliary equation.

What is a Second Order Differential Equation?

A differential equation is an equation that consists of a function and its derivative. A differential equation that consists of a function and its second-order derivative is called a second order differential equation. Mathematically, it is written as y'' + p(x)y' + q(x)y = f(x), which is a non-homogeneous second order differential equation if f(x) is not equal to the zero function and p(x), q(x) are functions of x. It can also be written as F(x, y, y', y'') = 0. Further, let us explore the definitions of the different types of the second order differential equation.

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)}$$

where P(x),Q(x) and f(x) are functions of x

Second Order Differential Equation Definition

A second order differential equation is defined as a differential equation that includes a function and its second-order derivative and no other higher-order derivative of the function can appear in the equation. It can be of different types depending upon the power of the derivative and the functions involved. These differential equations can be solved using the auxiliary equation. Let us go through some special types of second order differential equations given below:

Linear Second Order Differential Equation

A linear second order differential equation is written as $y'' + p(x)y' + q(x)y = f(x)$, where the power of the second derivative y'' is equal to one which makes the equation linear. Some of its examples are $y'' + 6x = 5$, $y'' + xy' + y = 0$, etc.

Homogeneous Second Order Differential Equation

A second order differential equation $y'' + p(x)y' + q(x)y = f(x)$ is said to be a second order homogeneous differential equation if $f(x)$ is a zero function and hence mathematically it of the form, $y'' + p(x)y' + q(x)y = 0$. Some of its examples are $y'' + y' - 6y = 0$, $y'' - 9y' + 20y = 0$, etc.

Non-homogeneous Second Order Differential Equation

A differential equation of the form $y'' + p(x)y' + q(x)y = f(x)$ is said to be a non-homogeneous second order differential equation if $f(x)$ is not a zero function. Some of its examples are $y'' + y' - 6y = x$, $y'' - 9y' + 20y = \sin x$, etc.

Second Order Differential Equation with Constant Coefficients

The differential equation $y'' + p(x)y' + q(x)y = f(x)$ is called a second order differential equation with constant coefficients if the functions $p(x)$ and $q(x)$ are constants. Some of its examples are $y'' + y' - 6y = x$, $y'' - 9y' + 20y = \sin x$, etc.

Second Order Differential Equation with Variable Coefficients

The differential equation $y'' + p(x)y' + q(x)y = f(x)$ is called a second order differential equation with variable coefficients if the functions $p(x)$ and $q(x)$ are not constant functions and are functions of x. Some of its examples are $y'' + xy' - y \sin x = x$, $y'' - 9x^2y' + 2e^xy = 0$, etc.

Solving Second Order Differential Equation

Now that we have understood the meaning of second order differential equation and their different forms, we shall proceed towards learning how to solve them. Here, we will focus on learning to solve 2nd differential equations with constant coefficients using the method of undetermined coefficients. First, let us understand how to solve the second order homogeneous differential equations.

Solving Homogeneous Second Order Differential Equation

A homogeneous second order differential equation with constant coefficients is of the form $y'' + py' + qy = 0$, where p, q are constants. To solve this, we assume a general solution $y = e^{rx}$ of the given differential equation, where r is any constant, and follow the given steps:

- Step 1: Differentiate the assumed solution $y = e^{rx}$, and find $y' = re^{rx}$, $y'' = r^2e^{rx}$, where r is an arbitrary constant.

- Step 2: Substitute the derivatives in the given differential equation y" + py' + qy = 0. We have $r^2e^{rx} + pre^{rx} + qe^{rx} = 0 \Rightarrow e^{rx}(r^2 + rp + q) = 0 \Rightarrow r^2 + rp + q = 0$, which is called the auxiliary equation or characteristic equation.
- Step 3: Solve the auxiliary equation $r^2 + rp + q = 0$ and find its roots r_1 and r_2.
  - If r_1 and r_2 are real and distinct roots, then the general solution is $y = Ae^{r}{}_1x + Be^{r}{}_2x$
  - If $r_1 = r_2 = r$, then the general solution is $y = Ae^{rx} + Bxe^{rx}$
  - If $r_1 = a + bi$ and $r_2 = a - bi$ are complex roots, then the general solution is $y = e^{ax}(A \sin bx + B \cos bx)$

Let us consider a few examples of each type to understand how to determine the solution of the homogeneous second order differential equation.

Example 1: Solve the 2nd order differential equation y" - 6y' + 5y = 0

Solution: Assume $y = e^{rx}$ and find its first and second derivative: $y' = re^{rx}$, $y'' = r^2e^{rx}$

Next, substitute the values of y, y', and y" in y" - 6y' + 5y = 0. We have,

$r^2e^{rx} - 6re^{rx} + 5e^{rx} = 0$

$\Rightarrow e^{rx}(r^2 - 6r + 5) = 0$

$\Rightarrow r^2 - 6r + 5 = 0 \rightarrow$ Characteristic Equation

$\Rightarrow (r - 5)(r - 1) = 0$

$\Rightarrow r = 1, 5$

Since the roots of the characteristic equation are distinct and real, therefore the general solution of the given differential equation is $y = Ae^x + Be^{5x}$

Example 2: Solve the second order differential equation y" - 8y' + 16y = 0

Solution: Assume $y = e^{rx}$ and find its first and second derivative: $y' = re^{rx}$, $y'' = r^2e^{rx}$

Next, substitute the values of y, y', and y" in y" - 8y' + 16y = 0. We have,

$r^2e^{rx} - 8re^{rx} + 16e^{rx} = 0$

$\Rightarrow e^{rx}(r^2 - 8r + 16) = 0$

$\Rightarrow r^2 - 8r + 16 = 0 \rightarrow$ Auxiliary Equation

$\Rightarrow (r - 4)(r - 4) = 0$

$\Rightarrow r = 4, 4$

Since the roots of the characteristic equation are identical and real, therefore the general solution of the given differential equation is $y = Ae^{4x} + Bxe^{4x}$

Example 3: Solve the second order differential equation 9y" + 12y' + 29y = 0

Solution: Assume $y = e^{rx}$ and find its first and second derivative: $y' = re^{rx}$, $y'' = r^2e^{rx}$

Next, substitute the values of y, y', and y'' in 9y'' + 12y' + 29y = 0. We have,

$9r^2e^{rx} + 12re^{rx} + 29e^{rx} = 0$

$\Rightarrow e^{rx}(9r^2 + 12r + 29) = 0$

$\Rightarrow 9r^2 + 12r + 29 = 0 \rightarrow$ Characteristic Equation

$\Rightarrow r = [-12 \pm \sqrt{(12^2 - 4 \times 9 \times 29)}]/(2 \times 9)$

$\Rightarrow r = (-2/3) \pm (5/3)i$

Since the roots of the characteristic equation are complex conjugates, therefore the general solution of the given second order differential equation is $y = e^{(-2/3)x}[A \sin (5/3)x + B \cos (5/3)x]$.

Solving Non-Homogeneous Second Order Differential Equation

To find the solution of Non-Homogeneous Second Order Differential Equation y'' + py' + qy = f(x), the general solution is of the form $y = y_c + y_p$, where y_c is the complementary solution of the homogeneous second order differential equation y'' + py' + qy = 0 and y_p is the particular solution of the non-homogeneous differential equation y'' + py' + qy = f(x). Since y_c is the solution of the homogeneous differential equation, we can determine its value using the methods that we discussed in the previous section. Now, to find the particular solution y_p, we can guess the solution depending upon the value of f(x). The table given below shows the possible particular solution y_p corresponding to each f(x).

f(x)	**y_p**
be^{ax}	Ae^{ax}
ax^n + (lower order powers of x)	$C_nx^n + C_{n-1}x^{n-1} + ... + C_0$
P cos ax or Q sin ax	A cos ax + B sin ax

In case, f(x) is of a form other than the ones given in the table above, then we can use the method of variation of parameters to solve the non-homogeneous second order differential equation. Also, if f(x) is a sum combination of the functions given in the table, then we can determine the particular solution for each function separately and then take their sum to find the final particular solution of the given equation. Let us now consider a few examples of second order differential equations and solve them using the method of undetermined coefficients:

Example 1: Find the complete solution of the second order differential equation $y'' - 6y' + 5y = e^{-3x}$.

Solution: To find the complete solution, first we will find the general solution of the homogeneous differential equation $y'' - 6y' + 5y = 0$.

We have solved this equation in the previous section in the solved examples (Example 1) and hence the complementary solution is $y_c = Ae^x + Be^{5x}$

Next, we will find the particular solution y_p. Since $f(x) = e^{-3x}$ is of the form be^{ax}, let us assume $y_p = Ae^{-3x}$. Now differentiating y_p, we have

$y_p' = -3Ae^{-3x}$ and $y_p'' = 9Ae^{-3x}$. Substituting these values in the given second order differential equation, we have

$y_p'' - 6y_p' + 5y_p = e^{-3x}$

$\Rightarrow 9Ae^{-3x} - 6(-3Ae^{-3x}) + 5Ae^{-3x} = e^{-3x}$

$\Rightarrow Ae^{-3x} (9 + 18 + 5) = e^{-3x}$

$\Rightarrow 32 A e^{-3x} = e^{-3x}$

$\Rightarrow A = 1/32$

Hence, the particular solution $y_p = (1/32) e^{-3x}$

Answer: Therefore, the complete solution of the given non-homogeneous 2nd order differential equation $y'' - 6y' + 5y = e^{-3x}$ is $y = Ae^x + Be^{5x} + (1/32) e^{-3x}$

Example 2: Solve the second order differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$

Solution: As we have solved the homogeneous differential equation $y'' - 6y' + 5y = 0$ in the previous section (Example 1), we have the complementary solution $y_c = Ae^x + Be^{5x}$

Next, we will find the particular solution of the given differential equation individually for $\cos 2x$ and e^{-3x}, that is, determine the particular solution of $y'' - 6y' + 5y = \cos 2x$ and $y'' - 6y' + 5y = e^{-3x}$ separately. From example 1 above, we have the particular solution of the differential equation $y'' - 6y' + 5y = e^{-3x}$ corresponding to e^{-3x} as $(1/32) e^{-3x}$. Now, we will find the particular solution of the equation $y'' - 6y' + 5y = \cos 2x$ using the table. Assume the particular solution of the form $Y_p = A \cos 2x + B \sin 2x$. Differentiating this, we have $Y_p' = -2A \sin 2x + 2B \cos 2x$

and $Y_p'' = -4A \cos 2x - 4B \sin 2x$. Substituting these values in the differential equation $y'' - 6y' + 5y = \cos 2x$, we have

$-4A \cos 2x - 4B \sin 2x - 6(-2A \sin 2x + 2B \cos 2x) + 5(A \cos 2x + B \sin 2x) = \cos 2x$

$\Rightarrow -4A \cos 2x - 4B \sin 2x + 12A \sin 2x - 12B \cos 2x + 5A \cos 2x + 5B \sin 2x = \cos 2x$

$\Rightarrow (A - 12B) \cos 2x + (B + 12A) \sin 2x = \cos 2x$

$\Rightarrow A - 12B = 1$ and $B + 12A = 0$

$\Rightarrow A = 1/145$ and $B = -12/145$

$\Rightarrow Y_p = (1/145) \cos 2x - (12/145) \sin 2x$

Now, taking the sum of both particular solutions, the final particular solution of the given second order differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$ is $y_p = (1/32) e^{-3x} + (1/145) \cos 2x - (12/145) \sin 2x$.

Answer: Therefore, the complete solution of the differential equation $y'' - 6y' + 5y = \cos 2x + e^{-3x}$ is $y = y_c + y_p = Ae^x + Be^{5x} + (1/32) e^{-3x} + (1/145) \cos 2x - (12/145) \sin 2x$

Second Order Differential Equation

- If y_1 and y_2 are two linearly independent solutions of the homogeneous second order differential equation $y'' + py' + qy = 0$, then the particular solution of the corresponding second order non-homogeneous differential equation $y'' + py' + qy = f(x)$ can be determined using the formula $y_p = -y_1 \int[y_2 f(x)/W(y_1, y_2)]\, dx + y_2 \int[y_1 f(x)/W(y_1, y_2)]\, dx$, where $W(y_1, y_2) = y_1y_2' - y_2y_1'$ is called the Wronskian. This method of finding the solution is called the method of variation of parameters.
- The method to find the solution of second-order differential equations with variable coefficients is complex and is based on guessing the solution.

The second-order linear differential equations with variable coefficients are differential equations whose coefficients are a function of a certain variable. A second-order linear differential equation has a general form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are functions of the independent variable x. If P and Q are some constant quantities, then the above equation is known as a second-order linear differential equation with constant coefficients.

If R = 0 then the equation is called a homogeneous linear differential equation of second order, otherwise it is non-homogenous.

CONCLUSION

Solution of Second-Order Differential Equations with Variable Coefficients

The solution of the second-order linear differential equation with variable coefficients can be determined using the Laplace transform. In particular, when the equations have terms of the form $t^m y^{(n)}(t)$, its Laplace transform is $(-1)^m\, d^m/ds[L\{y^{(n)}(t)\}]$.

Let us understand with an example, we have a second-order linear differential equation with variable coefficients

$ty'' + (1 - 2t)y' - 2y = 0$ where $y(0) = 1$ and $y'(0) = 2$

Taking Laplace transforms on both the side, we have

$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$

$$\Rightarrow -\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + [sL\{y\} - y(0)] + 2\frac{d}{ds}[sL\{y\} - y(0)] - 2L\{y\} = 0$$

Integrating both sides, we get

$\ln|z| + \ln|s - 2| = \ln C_1$

Or, $L\{y\} = C_1/(s - 2)$

Now, taking the inverse Laplace transform on both sides, we get

$y = C_1 L^{-1}\{1/(s - 2)\} = C_1 e^{2t}$

$\Rightarrow y = C_1 e^{2t}$

But $y(0) = 1$, therefore $C_1 = 1$

Hence, the solution of the given differential equation is $y = e^{2t}$.

General and Fundamental Solutions

Before defining the Fundamental and general solution of a second-order linear differential equation with variable coefficients, we must know about the Wronskian of functions.

The Wronskian of function $y_1, y_2: \mathbf{R} \to \mathbf{R}$ is the function defined by

$W_{y1y2}(t) = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)\ \forall\, t \in \mathbf{R}$

Alternatively,

$$W_{y_1y_2}(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Fundamental Solution

If y_1 and y_2 are two solutions of the differential equation $y'' + a_1(t)\, y' + a_o(t)\, y = 0$, then y_1 and y_2 are called fundamental solution if and only if y_1 and y_2 are linearly independent, that is, $W_{y1y2} \neq 0$.

General Solution

If y_1 and y_2 are two fundamental solution of the differential equation $y'' + a_1(t)\, y' + a_o(t)\, y = 0$, and c_1 and c_2 be any two arbitrary constants, then

$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$ is said to be the general solution of the given differential equation

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