SIMPLE AND EASIER METHOD TO FIND ANALYTIC FUNCTION

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Abstract: ***In this paper, I have developed a easier method to find the complex variable analytic function*** $f(z) = u(x,y) + iv(x,y)$ ***if any of*** u ***or*** v ***is given and named it 'KK's method'. A complex variable function*** $f(z) = u(x,y) + iv(x,y)$ ***is said to be analytic in a region R, if*** u ***and*** v ***are continuous and differentiable in the region R. There are several methods to find the analytic function if either*** u ***or*** v ***is known, such as 'Direct method', 'Milne-Thomson's method' and 'Exact-differential equation method'.***

Key words: ***Analytic function, Complex variable function, Cauchy- Riemann equation, Milne-Thomson's method, Exact- differential equation, Harmonic function, Laplace's equation, Holomorphic function, Meromorphic function, poles, essential singular point, single valued function, entire function.***

I. INTRODUCTION

Basic fact of the theory of analytic functions is the identity of the corresponding classes of functions in an arbitrary domain of the complex plane. An analytic function is the function that is locally given by a convergent series. There are real analytic function and complex analytic function. A complex variable function $f(z) = u(x,y) + iv(x,y)$ is said to be analytic in a region R, if both $u(x,y)$ and $v(x,y)$ are continuous and infinitely differentiable at each and every point of region R. The concept of analyticity in complex plane is based on the existence of derivative with respect to complex variable. In this paper, I will derive an easier way to find the analytic function $f(z)$ if either u or v is given.

II. HISTORY

A great contribution to number theory and to analytic mechanics had made by the Italian French mathematician ‗Giuseppe Luigi Lagrangia' His most important book (1788, ―Analytic Mechanics‖), is the base for all the later work in the field of analytic function. Gaspard Monge was a leading professor in mathematics of Ecole polytechnique. His lecture were published as (1797, ―Theory of analytic functions‖) and (1804, ―Lessons on the calculus of functions‖) were the first text book on the real analytic function. Some other important contributors in the field of real analytic function and complex analytic function are Euler, Gauss, Riemann, Cauchy and Weierstrass. In the field of complex analysis in mathematics, the Cauchy–Riemann equations, named after Augustin Cauchy and Bernhard Riemann, consist of a system of two partial differential equations which, together with certain continuity and differentiability criteria, form a necessary and sufficient condition for a complex function tobe complex differentiable that is Holomorphic. This system of equations first appeared in the work of Jean le Rond d'Alembert (d'Alembert 1752). Later, Leonhard Euler connected this system to the analytic functions (Euler 1797). Cauchy (1814) then used these equations to construct his theory of functions.

Riemann's dissertation (Riemann 1851) on the theory of functions appeared in 1851. Complex variables for analytic function have a very wide range to work on it. There were several work had done in later 19th century, earlier 20th century and it is going on.

III. APPLICATION

Theory of complex variable analytic function is used in several fields. It is very useful in solving boundary value problems. It is used in analysis of fluid flow, electrostatic field, magnetic field, heat flow and many more. Complex variable analysis is also used to find the definite integral, asymptotic solutions of differential equations and integral transform.

FLUID FLOW: For a given flow of an incompressible fluid there exist an analytic function

$$f(z) = \varphi(x,y) + i\psi(x,y)$$

$f(z)$ is called as complex potential of the flow, φ is called Velocity potential and ψ is called stream function.

ELECTROSTATIC FIELD: The force of attraction or repulsion between the charged particle is governed by Coulomb‘s law. This force can be expressed as the gradient of a function φ called the electrostatic potential. This electrostatic potential satisfies the Laplace‘s equation

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

Therefore φ will be the real part of some analytic function
$f(z) = \varphi(x, y) + i\psi(x, y)$

MAGNETIC FIELD: Let φ is a scalar magnetic potential. Then magnetic field intensity can be written as
$H = -\nabla\varphi$
From Maxwell‘s equation for scalar magnetic field

$$\nabla.B = 0$$
$$\nabla.(\mu H) = 0$$
$$-\mu\nabla.(\nabla\varphi) = 0$$
$$\nabla.(\nabla\varphi) = 0$$
$$\nabla^2\varphi = 0$$

Therefore the magnetic potential φ will be the real part of some complex variable analytic function
$f(z) = \varphi(x, y) + i\psi(x, y)$

HEAT FLOW: Laplace‘s equation governs heat flow problems that are steady and time independent.

Heat conduction in a body of homogeneous material is given by the equation

$$\frac{\partial T}{\partial t} = c^2\nabla^2 T$$

Here function T is temperature, t is time and c^2 is a positive constant.

For the steady flow $\frac{\partial T}{\partial t} = 0$, this reduces the heat equation to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$T(x, y)$ is called the heat potential. This is the real part of complex heat potential

$$f(z) = T(x, y) + i\psi(x, y)$$

IV. DEFINITIONS

COMPLEX ALGEBRA: A number given by the expression $z = x + iy$ is a complex number. Here $i = \sqrt{-1}$ is an imaginary number pronounced as _iota‘ x is called as real part of complex number and y is called as imaginary part of complex number.

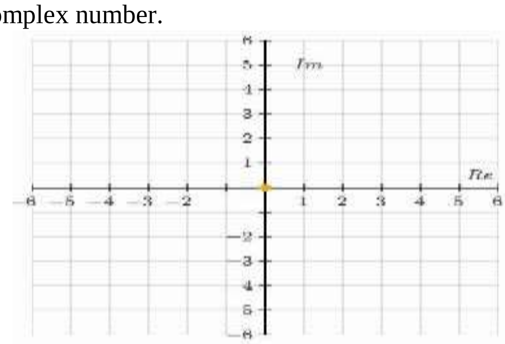

Figure 1: Complex plane

MODULUS (MAGNITUDE) OF A COMPLEX NUMBER: The Modulus or magnitude of a complex number $z = x + iy$ is the distance of the point $P(x, y)$ from the origin. It is denoted by $|z|$ Hence,

$$|z| = \sqrt{x^2 + y^2}$$

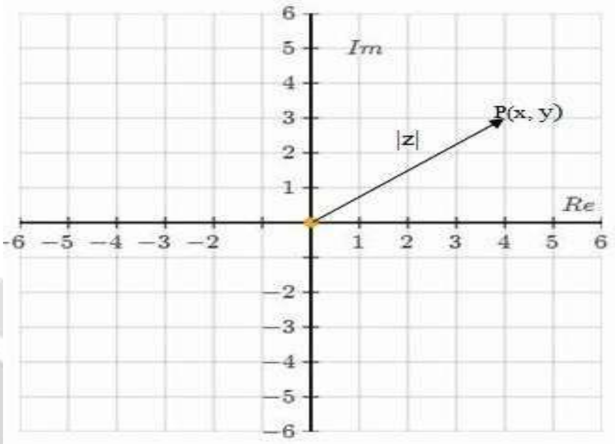

Figure 2: Modulus of a complex number

ARGUMENT (AMPLITUDE) OF A COMPLEX NUMBER: Argument of any complex number $z = x + iy$ is the tan of angle made by the line joining the point $P(x, y)$ to the origin from real axis.

Hence,

$$Arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

For the principal argument of z

$-\pi < arg(z) \le \pi$

Therefore,

If point $P(x, y)$ is in first quadrant

$$Arg(z) = \tan^{-1}\left|\frac{y}{x}\right|$$

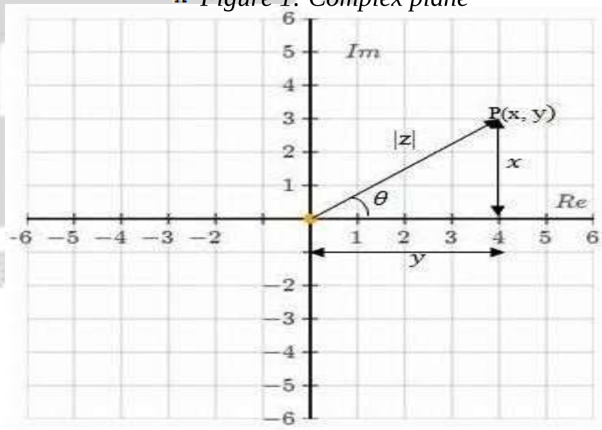

Figure 3: Argument of a complex number

If point $P(x, y)$ is in second quadrant

$$Arg(z) = \pi - \tan^{-1}\left|\frac{y}{x}\right|$$

If point $P(x, y)$ is in third quadrant

$$Arg(z) = -\pi + \tan^{-1}\left|\frac{y}{x}\right|$$

If point $P(x, y)$ is in fourth quadrant

$$Arg(z) = -\tan^{-1}\left|\frac{y}{x}\right|$$

CONJUGATE OF A COMPLEX NUMBER: Conjugate of a complex number $z = x + iy$ is the image of the point $P(x, y)$ in real axis. It is denoted by $\bar{z}$

Hence,

$$\bar{z} = x - iy.$$

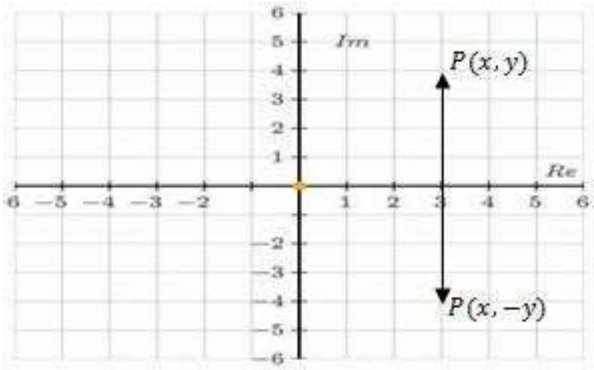

Figure 4: Conjugate of a complex number

COMPLEX VARIABLE FUNCTION: Functions of a complex variable provide us some powerful and widely useful tools in mathematical analysis as well as in theoretical physics. Wave function, AC impedance $z(\omega)$etc are some important complex variables functions. A complex variable function is given by $F(z) = u(x, y) + iv(x, y)$. Here, u and v both are the function of x and y .

ANALYTIC FUNCTION: A single valued function that possesses a unique derivative with respect to z at all points in a region R is called an Analytic function of z in that region.

A complex variable function $f(z) = u + iv$ is an analytic function in the region R if –

$f(z) = u + iv$, such that u, v, u_x, v_x, u_y and v_y are continuous and single valued function in x and y

It must satisfies the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

CAUCHY- RIEMANN EQUATIONS: A complex variable function $f(z) = u + iv$ is said to be analytic in region R, if it satisfies the relations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. These equations are called as Cauchy- Riemann equations.

HARMONIC FUNCTION: A function f is said to be harmonic if it satisfies the Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. A Complex variable function $f(z) = u(x, y) + iv(x, y)$ is said to Harmonic function if both u and v satisfies the Laplace's equation. Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Let $f(z) = u(x, y) + iv(x, y)$ is an analytic function.

From Cauchy- Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{... (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{... (2)}$$

Differentiating equation (1) with respect to x and equation (2) with respect to y and adding them

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

Therefore, every analytic function $f(z) = u + iv$represents a harmonic function.

EXACT DIFFERENTIAL EQUATION: A differential equation given by $dv = Mdx + Ndy$ is said to be in exact form if—

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of such differential equation is given by—

$$v = \int_{y=\text{constant}} Mdx + \int (\text{terms of N not containg x})dy$$

V. DERIVATIONS

DERIVATIVE OF A COMPLEX VARIABLE FUNCTION: Derivative of any function $f(z)$ is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

Let $f(z) = u + iv$ is any complex variable function with $z = x + iy$.

Hence, $\Delta z = \Delta x + i\Delta y$

$\Delta f = \Delta u + i\Delta v$

Assuming that partial derivative with respect to x and y exits for both the function u and v

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$\Delta y = 0$$

$$\Delta z \to 0 \Rightarrow \Delta x \to 0$$

Again, by considering

As

Hence,

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{... (3)}$$

Now, by considering $\Delta x = 0$

As $\Delta z \to 0 \Rightarrow \Delta y \to 0$

Hence,

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left(-i\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta y}\right)$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{... (4)}$$

From equations (1) and (2)—

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{... (5)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{... (6)}$$

Therefore, derivative of a complex variable function $f(z) = u + iv$ exists, if and only if equation (5) and (6) get satisfied.

DETERMINATION OF ANALYTIC FUNCTION: An analytic function $f(z) = u + iv$can be determined if any of u or v is given.

These are the methods to find the analytic function

* Direct method
* Milne-Thomson's method
* Exact differential equation method
* KK's method [A method developed by the author of this research paper]

DIRECT METHOD: Let $f(z) = u + iv$ is an analytic function.

Case-I If u is given then v can be finding out in following steps

Step-I Evaluate $\frac{\partial u}{\partial x}$

Step-II Evaluate $\frac{\partial v}{\partial y}$ $\left[\text{by C} - \text{R equation } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.\right]$

Step-III Integrate $\frac{\partial v}{\partial y}$ with respect to y to find v with taking integrating constant $f(x)$

Step-IV Differentiate v (from step-III) with respect to x $\left[\text{Evaluate } \frac{\partial v}{\partial x} \text{ containing } f'(x)\right]$

Step-V Evaluate $\frac{\partial u}{\partial y}$

Step-VI Evaluate $\frac{\partial v}{\partial x}$ $\left[\text{by C} - \text{R equation } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$

Step-VII Equate the values of $\frac{\partial v}{\partial x}$ from Step-IV and Step-VI and find $f'(x)$.

Step-VIII Integrate $f'(x)$ with respect to x to find $f(x)$.

Step-IX Now, substitute the value of $f(x)$ in Step-III to evaluate v

Case-II If v is given then u can be finding out in following steps

Step-I Evaluate $\frac{\partial v}{\partial y}$

Step-II Evaluate $\frac{\partial u}{\partial x}$ $\left[\text{by C} - \text{R equation } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.\right]$

Step-III Integrate $\frac{\partial u}{\partial x}$ with respect to x to find u with taking integrating constant $f(y)$

Step-IV Differentiate u (from step-III) with respect to y. $\left[\text{Evaluate } \frac{\partial u}{\partial y} \text{ containing } f'(y)\right]$

Step-V Evaluate $\frac{\partial v}{\partial x}$

Step-VI Evaluate $\frac{\partial u}{\partial y}$ $\left[\text{by C} - \text{R equation } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

Step-VII Equate the values of $\frac{\partial u}{\partial y}$ from Step-IV and Step-VI and find $f'(y)$.

Step-VIII Integrate $f'(y)$ with respect to y to find $f(y)$.

Step-IX Now, substitute the value of $f(y)$ in Step-III to evaluate u

MILNE-THOMSON'S METHOD: Let a complex variable function is given by $f(z) = u(x,y) + iv(x,y)$ ... (7)

Here $z = x + iy$.

Now, writing x and y in terms of z and $\bar{z}$

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2}$$

On differentiating $f(z)$ with respect to x

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x}$$

Now, by considering $z = \bar{z} \Rightarrow x = z$ and $y = 0$

Therefore,

$$\frac{\partial f(z)}{\partial z} = \frac{\partial u(z,0)}{\partial z} + i\frac{\partial v(z,0)}{\partial z}$$

On integrating

$$f(z) = u(z,0) + iv(z,0)$$

This equation is same if we will replace x with z and y with 0 in equation (7).

EXACT DIFFERENTIAL EQUATION METHOD: Let $f(z) = u + iv$ is an analytic function. Hence, u and v both will be a harmonic function.

Case-I If u is given

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{[From C-R equation]} \quad ... (8)$$

On comparing with $dv = Mdx + Ndy$

$$M = -\frac{\partial u}{\partial y} \text{ and } N = \frac{\partial u}{\partial x}$$

Here, $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Therefore $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ [$\because f(z)$ is a harmonic function]

Hence, equation (8) is in exact form. So, its solution is given by

$$v = \int_{y=constant} \left(-\frac{\partial u}{\partial y}\right)dx + \int \left(\text{terms of}\left(\frac{\partial u}{\partial x}\right) \text{ not containing } x\right)dy$$

Case-II If v is given then u can be finding out in following steps

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad \text{[From C-R equation]} \quad ... (9)$$

On comparing with $du = Mdx + Ndy$

$$M = \frac{\partial v}{\partial y} \text{ and } N = -\frac{\partial v}{\partial x}$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial N}{\partial x} = -\frac{\partial^2 v}{\partial x^2}$

Therefore $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right)$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ [$\because f(z)$ is a harmonic function]

Hence, equation (9) is in exact form. So, its solution is given by

$$u = \int_{y=constant} \left(\frac{\partial v}{\partial y}\right)dx + \int \left(\text{terms of}\left(-\frac{\partial v}{\partial x}\right) \text{ not containing } x\right)dy$$

KK'S METHOD: This is a unique method to find an analytic function in terms of z if any of u or v is given.

Case-I If u is given

Let $f(z) = u(x,y) + iv(x,y)$ ... (10)

is an analytic function.

Here, $z = x + iy$

On taking conjugate of z

$\bar{z} = x - iy$

Hence, $x = \frac{z+\bar{z}}{2}$ ... (11)

$y = \frac{z-\bar{z}}{2i}$ ... (12)

On taking the conjugate of $f(z)$

$\overline{f(z)} = \bar{f}(\bar{z}) = u(x,y) - iv(x,y)$ ... (13)

Adding equations (10) and (13)

$$f(z) + \bar{f}(\bar{z}) = 2u(x, y)$$
$$\Rightarrow f(z) = 2u(x, y) - \bar{f}(\bar{z})$$

Assuming that $\bar{z} = 0$, therefore from equations (11) and (12) $x = \frac{z}{2}$ and $y = \frac{z}{2i}$

Hence,

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0) \quad \text{... (14)}$$

Again, as $\bar{z} = x - iy = 0$. so, it can be taken as $x = 0, y = 0$

Now, from equation (13)

$$\bar{f}(\bar{z}) = u(x, y) - iv(x,y)$$
$$\Rightarrow \bar{f}(0) = u(0,0) - iv(0,0) \quad \text{... (15)}$$

As $v(x,y)$ is a function of x and y and this is not known. After substituting the value $x = 0$ and $y = 0$ in $v(x,y)$ it will give $v(0,0) = c$

Therefore, from equation (15)

$\bar{f}(0) = u(0,0) - ic$

On substituting the value of $\bar{f}(0)$ in equation (14)

$$\left[f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic\right]$$

The above expression is directly used to find the analytic function if $u(x, y)$ is given, provided that $u(0,0)$ and $v(0,0)$ exits.

Case-II If v is given

Let $f(z) = u(x, y) + iv(x, y)$ ... (16)

is an analytic function.

Here, $z = x + iy$

On taking conjugate of z

$\bar{z} = x - iy$

Hence, $x = \frac{z+\bar{z}}{2}$ ... (17)

$y = \frac{z-\bar{z}}{2i}$ ... (18)

On taking the conjugate of $f(z)$

$\overline{f(z)} = \bar{f}(\bar{z}) = u(x, y) - iv(x, y)$ ... (19)

Adding equations (10) and (13)—

$$f(z) - \bar{f}(\bar{z}) = 2iv(x,y)$$
$$\Rightarrow f(z) = 2iv(x,y) - \bar{f}(\bar{z})$$

Assuming that $\bar{z} = 0$, therefore from equations (17) and (18) $x = \frac{z}{2}$ and $y = \frac{z}{2i}$

Hence,

$$f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0) \quad \text{... (20)}$$

Again, as $\bar{z} = x - iy = 0$. so, it can be taken as $x = 0, y = 0$

Now, from equation (19)

$$\bar{f}(\bar{z}) = u(x, y) - iv(x,y)$$
$$\Rightarrow \bar{f}(0) = u(0,0) - iv(0,0) \quad \text{... (21)}$$

As $u(x, y)$ is a function of x and y and this is not known. After substituting the value $x = 0$ and $y = 0$ in $u(x, y)$ it will give $u(0,0) = c$

Therefore, from equation (21)

$\bar{f}(0) = c - iv(0,0)$

On substituting the value of $\bar{f}(0)$ in equation (20)

$$\left[f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) + iv(0,0) + c\right]$$

The above expression is directly used to find the analytic function if $v(x,y)$ is given, provided that $u(0,0)$ and $v(0,0)$ exits.

NOTE: This method is not applicable, if either $u(x, y)$ or $v(x,y)$ do not exit at $(0,0)$.

VI. CONCLUSION

KK's Method is much easier method to find an analytic function if either of u or v is given without using Cauchy-Riemann equation for complex variable. It cannot be used if $u(x, y)$ or $v(x, y)$ is not defined at $(0,0)$.

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