SOME ANALYTICAL METHODS FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

In recent years there has been an explosion of interest in the study of numerical methods for solving nonlinear Partial Differential Equations (PDEs) describing nonlinear wave phenomena with the results often motivated by the use of computers. These equations are difficult to analyze mathematically. The modeling has witnessed major progress in recent years in terms of new methods, modifications, resolved problems and the range of applications. A variety of analytical, asymptotic and numerical models have been developed over the past few decades to provide a framework for understanding the wave phenomena describing In this paper, we present a comprehensive literature review of shock waves and various semi-analytical/ numerical methods in finding solution. We also present limitations in the present research work and scope for future work..

Keyword: - *Analytical method, nonlinear PDEs, nonlinear waves, Semi-analytical method*

1. INTRODUCTION

With the rapid development of nonlinear science, the development of numerical techniques for solving nonlinear equations is a subject of considerable interest. Numerical calculation of shock wave motion continues to be an important and difficult problem. Various computational methods have been used to find solution of equations describing nonlinear gas dynamic waves. Many numerical methods and linearization techniques coupled with basic discrtetization techniques such as finite difference (FDM), finite element(FEM) and finite volume methods(FVM) are used to find better approximate solution. While FDM involves the grid generation and replacement of derivatives with finite differences, FEM involves mesh generation and piecewise polynomial approximations and FVM is suitable for three dimensional domain and used for three-dimensional shock propagation. These techniques involve the discretization of time and space depending upon initial and boundary conditions. Different types of non linear gas dynamics equations have been solved by the researchers by applying various kinds of analytical and numerical methods. There are various mathematical approaches for existence, uniqueness and stability of shock wave solution such as Hamilton-Jacobi theory, viscosity method, non linear semigroup theory and method of c haracteristics. The basic method used for the calculation of the shock wave propagation is the method of characteristics, which does not include dissipation of energy. The most important numerical methods for solving the model equation, the Burgers equation are Godunov's method (1959) and Glimm's method. There are various methods that have been developed in the last couple of decades for extraction of shock wave solutions. These methods have undergone several modifications made by various researchers in an attempt to improve the accuracy or to expand the applications of the original methods. Some of the methods for shock wave solution apart from the basic methods are inverse scattering transform applied by Mark J. Ablowitz, Douglas E. Baldwin(2013)[1] to analyze the long-time asymptotic solution of the Korteweg–deVries equation for general, step-like initial data, Chisnell's method of approximate analytic solution of the converging shock wave problem by V. S. Kozhanov, I. A. Chernov (2011)[2], etc. Methods like Adomian Decomposition Method(ADM), Implicit and Explicit Difference Method, Homotopy Analysis Method (HAM) ,Optimal Homotopy Asymptotic Method, Homotopy Perturbation Method (HPM) Green Function and Least-Squares Spectral Method, Differential Transform

Method(DTM), Laplace Decomposition Method (LDM), etc., have been developed recently. From experiments with different numerical schemes for non-genuinely nonlinear systems, it was observed that certain schemes produce classical solutions while others give non-classical ones.

2. NONLINEAR PDES CONSIDERED UNDER SEMI-ANALYTICAL/NUMERICAL METHODS

Some of the nonlinear PDEs considered by various researchers for the solution by numerical/semi-analytical methods in this paper are as below.

- $u_{tt} + au_{xx} + bu + cu^p + du^{2p-1} = 0.$
- Modified KdV- Burgers Equation, $u_t + (u^3)_x = \alpha u_{xx} + \beta u_{xxx}$.
- Combined KdV- mKdV equation: $u_t + 6uu_x \pm 6u^2u_x + u_{xxx} = 0.$
- Fifth order KdV equation: $u_t + uu_x - uu_{xxx} - u_{xxxxx} = 0,\ with\ u(x,0) = e^x$.
- $u_{tt} = \frac{\partial^2}{\partial x \partial y}\left(u_{xx}\, u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(x\, y\, u_x\, u_y\right) - u$.
- $u_t + u_x = 2\, u_{xxt}\ , x \in R, t > 0, u(x,0) = e^{-x}$.
- $Gas\ dynamic\ equation\ \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = u(1-u),\ x \geq 0\ t \geq 0$, with initial condition u(x,0) = e^{-x}.
- Gas dynamic equation of the form $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = u(1-u) + g(x,t), 1 \geq x \geq 0\ , t > 0,\ with\ initial\ conition\ u(x,o) = e^{-x}$.
- $\frac{\partial^2 u(x,t)}{\partial t^2} + u^2 - u_x^2 = 0,\quad t > 0, u(x,0) = 0, u_t(x,0) = e^x$.
- Degasperis-Procesi equation of the form $u_t - u_{xxt} + 4uu_x - 3uu_{xx} - uu_{xxx} = f(u, u_x, u_t, u_{xx}, u_{xxx}, u_{xxt})$.
- Newell–Whitehead–Segel equation of the form $u_t\ (x,t) = ku_{xx} + a\, u(x,t) - bu^q(x,t)$.
- Radial diffusivity equation subject to various initial conditions of the form $\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{\partial f}{\partial t}$.
- $\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$, (x, t) $\in \mathbb{R} X \left[0, \frac{1}{2}\right), u(x,0) = 2x$.
- Gardner-KP equation: $(u_t + 6uu_x \pm 6u^2u_x + u_{xxx})_x + u_{yy}$=0.
- Whitham-Broer-Kaup PDE: $u_t + uu_x + v_x + \beta u_{xx} = 0,\quad v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx}$.

2.1 Homotopy Analysis Method (HAM)

The HAM, a semi-analytical technique for solving nonlinear PDEs, does not involve discretization of the variables and does not need small parameters in the equations. That means, HAM is valid for a nonlinear problem where small/large physical parameters are involved and it is easy to adjust and control the convergence radius of solution series by HAM. Thus, one can expect an accurate approximation by means of the HAM. The method was first described in 1992 by Liao for a given nonlinear differential equation called zeroth-order deformation equation. This method is known as an early HAM. But due to its general non convergence of approximate series of nonlinear equations, the method was further modified in 1997(normal HAM) to introduce a non-zero auxiliary convergence-control parameter to construct a two-parameter family of equations. To avoid the time consuming computation, Liao developed in 2010 an optimal HAM with only three convergence-control parameters and in 2012, Liao [3] suggested a generalized optimal HAM.

HAM has been successively applied to solve different types of nonlinear gas dynamic wave's problems. Song L, Zhang H(2007) [4] has applied this method to solve fractional KdV–Burgers–Kuramoto equation, Hossein Jafari, et.al (2009)[5] have applied it to homogeneous gas dynamic equation, Abbasbandy et.al(2011)[6] proposed the predictor HAM(PHAM) to predict the multiplicity of solutions of nonlinear equations. There have been recent useful modifications in terms of methods based on HAM developed by Liao(2012)[7] like, Optimal Homotopy Asymptotic Method, the Spectral HAM, the Generalized Boundary Element Method, and the Generalized Scaled Boundary FEM for solving variety of nonlinear PDEs.

2.2 Adomian decomposition method (ADM)

ADM was introduced and developed by George Adomian in 1986 and is well addressed in the literature. The ADM is semi- analytic method, requiring neither linearization nor perturbation. The procedure of ADM includes the steps like, separating the given PDE into linear and nonlinear terms, expressing unknown function(solution)in the series form, say $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$,decomposing the nonlinear function in terms of Adomian's polynomials($A_n(u)$), and finding the successive terms of the series solution, u_1, u_2, $u_{3,...}$ etc by recurrence relation using $A_n(u)$.ADM is considered to be a special case of HAM. During recent years, several researchers have tried to modify the ADM for shock wave solution. The modifications arise from overcoming the specific difficulties for the type of problem under consideration. D. Kaya and M. Inc(1999)[8] used this method to obtain an analytic solution of the nonlinear parabolic equation with initial conditions by employing the phenomena of the self- cancelling "noise" terms, Evans and Bulut (2002) [9], to solve homogeneous gas dynamics equation with initial and boundary conditions, Fathi Mallan, Kamel Al- Khaled(2006)[10], have applied this technique, to find an approximation of the analytic solution of the fully developed shocks which is valid for $0 \leq t < \infty$. S. Abbasbandy, M . T . Darvishi (2005) [11] used modified ADM for calculating a numerical solution of the one-dimensional Burger's equation where time discretization, instead of Hopf–Cole transformation has been used in decomposition, an efficient modification, two-step ADM (TSADM) by Luo XG(2005)[12], a method involving some efficient numerical algorithms to solve a system of two nonlinear equations with two variables based on Newton's method by Abbasbandy S(2005)[13], Jafari H et.al.(2006)[14] used MADM for system of nonlinear equations, Mehdi Dehghan et. al (2007)[15] used modified ADM–PADE technique, for solving homogeneous (inhomogeneous) two-dimensional parabolic equations, namely coupled Burger's equation to get solution with faster convergence rate and higher accuracy.

2.3 Homotopy Perturbation method (HPM)

The HPM is a semi analytical method which is a coupling of the traditional perturbation method and homotopy in topology. This method is a special case of HAM introduced by Ji-Huan He in 1988**.** The method has rapid convergence of solution series in most the cases with the help of only iteration. The researchers have used HPM to solve various types of nonlinear gas dynamic equations [16]. HPM was applied to solve coupled system of reaction-diffusion equation, fifth-order KdV equation by Ganji et al(2006), travelling wave solution of the KdV equation is obtained by Ozis and Yildirim(2007). Odibat and Momani developed a modified HPM (2008) to solve nonlinear PDEs of fractional order and quadratic Riccati differential equation of fractional order. This was applied to solve fractional gas dynamics equation and fractional heat and wave-like equations by Singh et al.(2013). Belendez et al. (2009) did a good investigation on this method. J. H. He (2014) [17] suggested a HPM with two expanding parameters which is especially effective for a nonlinear equation with two nonlinear terms, which might have different effects on the solution. Asma Ali Elbeleze et.al. (2014)[18] extended the applications of HPM to obtain approximate solution of fractional PDEs such as Burgers' equation of fractional order and fractional fourth-order parabolic PDE and obtained the convergence of the method. Khan and Wu (2011) [19] proposed the homotopy perturbation transform method (HPTM) for solving the nonlinear equations. Jagdev Singh, Devendra Kumar and Sushila(2012)[20] applied HPM and Laplace transform to solve gas dynamic equation. Prem Kiran G. Bhadane, and V. H. Pradhan (2013)[21] implemented ELzaki transform HPM(ETHPM) to obtain the approximate analytical solution of the nonlinear homogeneous and non-homogeneous gas dynamics equations of the form $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = u(1-u),\ x \geq 0\ t \geq 0$, with the initial condition $u(x,o) = e^{-x}$.

2.4 Variation iteration method (VIM)

VIM, introduced by Ji- Huan He (1999) [22], is based on general Lagrange multiplier and is different from the other non-linear analytical methods, such as perturbation methods. This method does not depend on small parameters. So it has wide range of applications in non-linear problems without linearization or small perturbations.

VIM has been applied to various nonlinear gas dynamic problems. A review about major applications of the method to nonlinear problems like, wave equation, fractional differential equations, oscillations and nonlinear problems arising in various engineering applications was given by Ji-Huan He and Xu-Hong Wu (2007) [23]. This method consists of a new iteration formulation and a useful formulation for determining approximately the period of a nonlinear oscillator. Abdou, MA, Soliman, AA(2005)[24] applied this method for solving Burger's and coupled Burger's equations, M. Matinfar et.al.(2009) [25] applied it to solve the nonlinear Whitham-Broer-Kaup(WBK) PDEs, Olayiwola et.al. (2011)[26], for solving gas dynamic equation of the form $\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}=u(1-u)+g(x,t), 1\geq x\geq 0, t>0, u(x,o)=e^{-x}$, by using Modified Variational Iteration method (MVIM). Qian Lijuan, et. al (2014)[27] applied for solving the generalized Degasperis-Procesi equation of the form $u_t-u_{xxt}+4uu_x-3uu_{xx}-uu_{xxx}=f(u,u_x,u_t, u_{xx},u_{xxx},u_{xxt})$ with suitable initial conditions. There are various improvements of this method such as A Step Variational Iteration Method(2013) and semi-inverse variational principle etc. Birol EbiG and Mustafa Bayram (2014)[28] implemented the fractional variational iteration method (FVIM), to obtain the analytical approximate solution in the form of convergent series, of time-fractional advection-dispersion equation (FADE) involving Jumarie's modification of Riemann-Liouville derivative. M. Matinfar, et.al, (2011)[29] implemented VIM coupled with He's polynomials (VIMHP) for finding the solution of homogeneous (non homogeneous) gas dynamic equation of the form $\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}=\square(1-u)+g(x,t), 1\geq x\geq 0, t>0$,by developing correct functional and calculating the Lagrange multipliers via variational theory.

2.5 Differential transform method (DTM)

DTM is an iterative procedure introduced by Zhou (1986) in a study of electric circuits. This method uses Taylor series for the solution of differential equations in the form of polynomials. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The differential transform is defined as follows (Erturk and Momani (2007), Ayaz (2004), Kangalgil and Ayaz (2009) and Yang et al. (2006)).

The differential transform of the k^{th} derivative of a function u(x) is defined as $U(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}u(x)\right]$ and the inverse transform of *U(k)* is defined as $u(x)=\sum_{k=0}^{\infty}\mathrm{U(k)}(x-x_0)^k$

Alquran and Al-Khaled (2010), Alquran and Al-Khamaiseh (2010), Odibat (2008), and Ayas (2004) gave useful discussions of various definitions, elementary properties and some theorems of the DTM that are used to solve nonlinear equations. The general procedure of DTM is to convert the given PDE to an ordinary differential equation (ODE) and then applying it to solve the given equation. Ravi-Kanth, A. S. V. and Aruna (2009)[30] have applied this method to solve nonlinear Klein-Gordon equation, M. A. Mohamed (2010) [31] compared DTM and ADM to obtain the numerical solutions for the system of dispersive long-wave equations (DLWE) in (2+1)-dimensions, where they showed these methods can be alternative ways for solution of the linear and nonlinear higher-order initial value problems. Ravi-Kanth, Abazari R, Ganji M (2011) [32] extended two-dimensional DTM and its application on nonlinear PDEs with proportional delay, Marwan Taiseer Alquran (2012)[33] applied DTM to obtain approximate solutions to nonlinear PDEs by considering three equations, namely, Benjamin-Bona-Mahony (BBM), Cahn-Hilliard equation and Gardner equation.

Several authors have proposed a variety of modifications to DTM. One of the modifications of DTM to solve gas dynamic equations is Reduced Differential Transform Method(RDTM) introduced by Y. Keskin (2009)[34-35]. Abazari R and Abazari M(2012)[36] presented a numerical simulation of generalized Hirota-Satsuma coupled KdV equation by the process of comparison of RDTM and DTM. A. Saravanan, N. Magesh(2013)[37] did a comparative study between the RDTM and ADM by handling the Newell–Whitehead–

Segel equation of the form $u_t\ (x,t) = \ ku_{xx} + \ a\,u(x,t) - \ bu^q(x,t)$, where he showed that the RDTM has an advantage over the ADM as it takes less time to solve the nonlinear problems without using the Adomian polynomials. Al-Amr, Mohammed O.(2014)[38] applied this for solving two types of nonlinear PDEs, namely, generalized Drinfeld-Sokolov (gDS) equations and Kaup-Kupershmidt (KK) equation. Benedict Iserom Ita (2014) [39], applied this to solve the radial diffusivity equation subject to various initial conditions of the form $\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{\partial f}{\partial t}$.

2.5 Laplace Decomposition Method (LDM)

The LDM a semi analytical method uses Laplace transform to solve nonlinear gas dynamic equations and is very well suited to physical problems where linearization, perturbation etc are not involved. LDM is based on Laplace transform and ADM. LDM was first proposed by Suheil A.Khuri (2001) and subsequently developed by Yusufoglu (2006) which has been successfully used to find the solution of differential equations. To overcome the difficulties of generation of noise term for inhomogeneous equations, M. Hussain developed a modified Laplace decomposition method (MLDM) which has rapid convergence of the series solution. Agadjanov solved Duffing equation with the help of this method. Elgasery applied LDM for the solution of Falkner-Skan equation where MLDM is implemented to three nonlinear PDEs. Another reliable modification of the LDM has been done by Khan (2009). Khan, Y., Faraz, N.(2010)[40] applied MLDM to obtain series solutions of the boundary layer equation and to solve boundary layer equation on semi-infinite domain. Many researchers combined LDM with other methods to obtain more accurate solution. Some of them are with HPM by Madani and Fathizadeh(2010), Mohyud-Din et al.(2010), Yildirim(2010),and with VIM by Hesameddini and Latifzadeh(2009), He et al.(2010). ZHU Hai-Xing, et al (2014)[41] used this method to solve nonlinear PDE with nonlinear term of any order PDE of the type $u_{tt} + au_{xx} + \ bu + cu^p + du^{2p-1} = 0$. Three distinct generalized numerical solutions of the equation were obtained which are the doubly periodic numerical solutions, kink-profile and bell-profile solutions. ADM and LTM are used in [42] to construct the solution of an initial boundary value convection-diffusion-dissipation equation in one-dimension and obtained exact solution.

Implementation of numerical/semi-analytical methods discussed above to solve nonlinear phenomena depends on different material media and flow parameters. For example, gas waves of high specific heat which admit mixed non linearity, propagation through relaxing gas or stratified atmosphere, viscid, inviscid, under magnetic field, of radiating gas, one-dimensional, two-dimensional three - dimensional, etc. The extensive work where high accuracy results and alternative approaches for the investigation of implosion problem is carried out by many researchers and can be found in the papers by Madhumita and Sharma [43], Whitham[44] and many other. Literature upto 2007 was provided by Abdolrahman Razani(2007) [45] along with Shock Waves in Gas Dynamics, Surveys in Mathematics and its Applications.

3. CONCLUSIONS

As the superposition principle cannot be applied for nonlinear gas dynamic waves and there are no general analytical methods for their solution, they are more difficult to analyze mathematically. So, a major challenge is to find a suitable asymptotic/numerical method to see how the wave evolves. Direct experimental detection of the shock wave is complicate because of its short duration. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer memory and time. There are issues of discontinuities in finite time regarding the model equation, the Burger's equation. These issues have been addressed mainly from theoretical perspective. Solutions of some of the PDEs are valid in infinite interval and for one dimensional domain while practical problems require the solution in finite interval. It is challenging to explore approximate theorems for three-dimensional shock propagation. The solutions of semi-analytical methods like ADM are valid in either in time or space problem domain. So, there is no scope for verification of the given boundary conditions. Thus, ADM cannot ensure the convergence of its approximation series. Many perturbation approximations are valid only for small physical parameters. So, it is not guaranteed that a perturbation result is valid in the whole region of all physical parameters. All traditional non-perturbation methods cannot guarantee the convergence of approximation series. In applying the semi-analytical methods/numerical methods for finding the analytical solutions, it has been noticed that the initial conditions considered for given nonlinear PDEs were generally simple standard functions like, x, e^{-x} etc. It is challenging to see whether the methods result in analytical solution or not, when complex initial conditions (dependent upon the physical relevance) are taken.

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