SOME PROBLEMS ABOUT ARITHMETIC MEAN AND HARMONIC MEAN

Pham Thi Linh[1], Tran Thi Mai[2], Nguyen Quynh hoa[3], Nguyen Thi Trang[4]

[1] *Thai Nguyen University of Economics and Business Administration , Thai Nguyen, Viet Nam*
[2] *Thai Nguyen University of Economics and Business Administration , Thai Nguyen, Viet Nam*
[3] *Thai Nguyen University of Economics and Business Administration , Thai Nguyen, Viet Nam*
[4] *Ha Long University, Quang Ninh, Viet Nam*

ABSTRACT

In mathematics, functional equations are very difficult and arise all areas of mathematics, even more, science, engineering, and social sciences. They appear at all levels of mathematics. The theory of functional equations were born very early. Many authors studied functional equations. In this artical, we study some problems about arithmetic mean and harmonic mean.

Keyword: *Functional equalities, arithmetic mean , harmonic mean.*

---

1. PRELIMINARIES

In this artical, we would like to look at some functional equations

$$\frac{x+y}{2};\ x,y\in\mathbb{R};$$

$$\sqrt{\frac{x^2+y^2}{2}};\ x,y\in\mathbb{R}^+;$$

$$\sqrt[n]{\frac{x^n+y^n}{2}};n=1,2,...;x,y\in\mathbb{R}^+;$$

$$\frac{2f(x)f(y)}{f(x)+f(y)},x,y\in\mathbb{R}.$$

In this paper, we use method of substitution

+) Example, let $x=u$ such that $f(u)$ appears much in the equation.

+) Let $x=u,y=v$ interchange to refer $f(u)$ and $f(v)$.

+) Let $f(0)=a,f(1)=b$.

+) If f is surjection, exist $a:f(a)=0$. Choice x,y to destroy $f(g(x,y))$ in the equation. The function has x, we show that it is injective or surjection.

+) To occur $f(x)$.

+) $f(x)=f(y)$ for all x, y ∈ A. Hence $f(x)=const$ for all $x\in A$.

On the other hand, we use math induction and the continuos of function.

2. SOME PROBLEMS ABOUT ARITHMETIC MEAN AND HARMONIC MEAN

In this part, we would like to look at some problems about arithmetic mean and harmonic mean.

2.1 Problem 1. Determine all functions $f(x)$ which are continuous on $\mathbb{R}^+$ and satisfy the equation

$$f\left(\frac{x+y}{2}\right)=\frac{2f(x)f(y)}{f(x)+f(y)},\forall x,y\in\mathbb{R}.\quad(1)$$

Solution. By assumption, we have

$$(1) \Leftrightarrow f\left(\frac{x+y}{2}\right)=\frac{2}{\frac{1}{f(x)}+\frac{1}{f(y)}}, \forall x, y \in \mathbb{R},$$

or

$$\frac{1}{f\left(\frac{x+y}{2}\right)}=\frac{\frac{1}{f(x)}+\frac{1}{f(y)}}{2}, \forall x, y \in \mathbb{R}.$$

Setting

$$g(u)=\frac{1}{f(u)}, \forall u \in \mathbb{R}.$$

We have

$$g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}.$$

Hence,

$$g(u)>0, \forall u \in \mathbb{R}.$$

By Problem 1 [1], we get

$$g(u)=au+b, \forall u \in \mathbb{R}.$$

Hence,

$$f(u)=\frac{1}{au+b}.$$

We chose a, b such that $f(x)$ is continuous and positively $\forall u \in \mathbb{R}$. Because the demominator is never zero then $a=0, b>0.$ So $f(u)=\frac{1}{b}.$

We can check directly $f(u)=\frac{1}{b}$ satisfies (1).

There for,

$$f(x) \equiv c; \text{ with for arbitrary } c>0.$$

2.2 Problem 2. Determine all functions $f: \mathbb{R} \to \mathbb{R}$ which are continuous on $\mathbb{R}$ and satisfy the equation

$$f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\frac{2f(x)f(y)}{f(x)+f(y)}, \forall x, y \in \mathbb{R}. \quad (2)$$

Solution. By assumption, we have $f(x) \neq 0, \forall x, y \in \mathbb{R}$. So

$$(2) \Leftrightarrow \frac{1}{f\left(\sqrt{\frac{x^2+y^2}{2}}\right)} = \frac{f(x)+f(y)}{2f(x)f(y)}, \forall x, y \in \mathbb{R}.$$

Setting

$$g(x) = \frac{1}{f(x)}, \forall x \in \mathbb{R}.$$

We have $g(x) \neq 0, \forall x \in \mathbb{R}$, $g(x)$ is continuous on $\mathbb{R}$, and

$$g\left(\sqrt{\frac{x^2+y^2}{2}}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}. \quad (2)$$

By Problem 2 $[1]$, we get

$$g(u) = au^2 + b, \forall u \in \mathbb{R}.$$

Since $g(x) \neq 0, \forall x \in \mathbb{R}$ then $a, b \geq 0$, and $b \neq 0$.

Hence

$$f(x) = \frac{1}{ax^2+b}, a, b \in \mathbb{R} : a, b \geq 0, b \neq 0.$$

We can check directly $f(x) = \frac{1}{ax^2+b}, a, b \in \mathbb{R} : ab \geq 0, b \neq 0$ satisfies problem.

Hence,

$$f(x) = \frac{1}{ax^2+b}, a, b \in \mathbb{R} : a, b \geq 0, b \neq 0.$$

2.3 Problem 3. Determine all functions $f(x)$ which are continuous on $\mathbb{R}$ and satisfy the equation

$$f\left(\sqrt[n]{\frac{x^n+y^n}{2}}\right) = \frac{2f(x)f(y)}{f(x)+f(y)}, \forall x, y \in \mathbb{R}, n = 1, 2.... \quad (3)$$

Solution. By assumption, we have $f(x) \neq 0$. So

$$(3) \Leftrightarrow \frac{1}{f\left(\sqrt[n]{\frac{x^n+y^n}{2}}\right)} = \frac{f(x)+f(y)}{2f(x)f(y)}, \forall x, y \in \mathbb{R}.$$

Setting $g(x) = \frac{1}{f(x)}, \forall x \in \mathbb{R}$, we have $g(x) \neq 0, \forall x \in \mathbb{R}$, $g(x)$ is continuous on $\mathbb{R}$, and

$$g\left(\sqrt[n]{\frac{x^n+y^n}{2}}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}.$$

By Problem 3 $[1]$, we get

$$g(x) = ax^n + b, \forall x \in \mathbb{R}.$$

Since $g(x) \neq 0, \forall x \in \mathbb{R}$ then $a, b \geq 0,$ and $b \neq 0$.
Hence

$$f(x) = \frac{1}{ax^n + b}, a, b \in \mathbb{R} : a, b \geq 0, b \neq 0.$$

We can check directly $f(x) = \frac{1}{ax^n + b}, a, b \in \mathbb{R} : a, b \geq 0, b \neq 0$ satisfies problem.

Hence,

$$f(x) = \frac{1}{ax^n + b}, a, b \in \mathbb{R} : a, b \geq 0, b \neq 0 .$$

3. CONCLUSIONS

In this paper, we establish some problems about arithmetic mean and harmonic mean. It is very good for teachers and students.

4. REFERENCES

[1]. Pham Thi Linh, Nguyen Thi Thu Hang, Nguyen Thi Trang (2020), *The ralationship between quadratic average, arbitrary level average and arithmetic average* , IJARIIE, India

[2]. Pl. Kannappan (2000), *Functional equations and inequalities with applications*, Springer, Monogaphs in Mathematics.

[3]. M. Kuczma, B. Choczewski, R. Ger (1990), *Interative functional equations*, Cambridge University Press, Cambridge/New York/Port Chester/Melbourne/Sydney.