STUDY AND DETERMINATION OF THE NATURAL FREQUENCIES AND NATURAL MODES OF BENDING VIBRATION OF CARBON/EPOXY COMPOSITE TUBE AND STEEL TUBE

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ABSTRACT

The aim of this work is to determine the behavior of two beams made with different materials and subjected to bending vibrations. The study was carried out on two tubes with non-negligible masses, with constant circular section, with the same geometric characteristics, and one of which is in carbon/epoxy composite and the other in steel. We consider usual types of connections for the two ends of each tube such as: two supported ends, two fixed ends, one fixed end and the other free. The results of the study have clearly shown that the type of end connections and the nature of the material have a great influence on the natural frequencies of the two tubes studied. Indeed, the tube with fixed ends has higher natural frequencies than those obtained from the other types of connection of the ends. The values of the natural frequencies of the Carbon/Epoxy tube are higher than those of the steel tube. In addition, the natural modes depend only on the type of connection.

Keywords : *vibration, bending, tube, frequency, mode, composite, steel*

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1. INTRODUCTION

A straight beam is one of the elements constituting a mechanical construction which can be industrial machines. But during the operation of these machines, there are times when the bending vibrations of these elements appear. However, the amplitudes of the movements of these constituents are very large if they vibrate at their own frequencies, which can cause the destruction of these elements.
Therefore, it is very important to know the natural frequencies of a beam and to avoid them so as not to encounter the phenomenon of resonance.

This study is carried out on two beams which are tubes with non-negligible masses, with constant circular section, unloaded and made with two different materials. The first tube is in Carbon/Epoxy composite while the second is in steel.
The types of connections considered for the ends of a tube are: two supported ends, two fixed ends, one fixed end and the other free.
The purpose of this research work is to determine the influence of the nature of the material and the type of end connections on the natural frequencies and natural modes of the studied tubes subjected to bending vibrations.

2. NATURAL FREQUENCIES AND NATURAL MODES OF A BEAM IN BENDING VIBRATION

We consider an element of mass dm whose length is dx of a beam whose length is L.

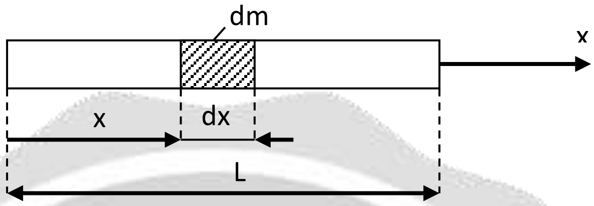

Fig-1 : Schematic representation of the beam

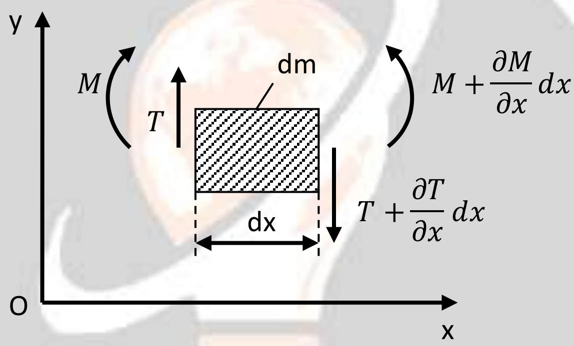

Fig-2 : Forces applied to the beam element

The movement of the beam element referred to a system of axes (Ox, Oy) is defined by :

T : Shear force
M : Bending moment
ρ : Volumic mass
dm : Mass of the element
S: Area of the cross section
I : Quadratic moment of the cross section
E : Modulus of elasticity
L : Length of the beam
y : Arrow
t : Time

The application of the fundamental principle of dynamics leads to

$$T - T - \frac{\partial T}{\partial x} dx = dm \frac{d^2y}{dt^2} = \rho S \, dx \frac{d^2y}{dt^2}$$

$$\frac{\partial T}{\partial x} = -\rho S \frac{d^2y}{dt^2} \quad (1)$$

In resistance of materials, the differential equation of the deformation is

$$M = EI \frac{d^2y}{dx^2} \quad (2)$$

Or $$T = \frac{dM}{dx} \tag{3}$$

$$\frac{\partial T}{\partial x} = \frac{d^2 M}{dx^2} = EI\frac{d^4 y}{dx^4}$$

(1) becomes

$$EI\frac{d^4 y}{dx^4} = -\rho S\ \frac{d^2 y}{dt^2} \tag{4}$$

$$EI\frac{d^4 y}{dx^4} + \rho S\ \frac{d^2 y}{dt^2} = 0 \tag{5}$$

Note that y is a function of x and t.
We put

$$y(x,t) = \phi(x).f(t) \tag{6}$$

This leads to

$$\frac{\partial^4 y}{\partial x^4} = \frac{d^4\phi(x)}{dx^4}.f(t) \tag{7}$$

$$\frac{d^2 y}{dt^2} = \phi(x).\frac{d^2 f(t)}{dt^2} \tag{8}$$

By grouping in one member the term as a function of x and in the other that as a function of t, the relations (4) , (7) and (8) gives

$$\frac{EI}{\rho S}\frac{d^4\ \phi(x)}{dx^4}.\frac{1}{\phi(x)} = -\frac{1}{f(t)}.\frac{d^2 f(t)}{dt^2} = \omega^2 \tag{9}$$

Here we have the equality of two functions of two different variables. This equality is equivalent to a constant ω^2. This allows us to write that

$$\frac{EI}{\rho S}\frac{d^4\ \phi(x)}{dx^4}.\frac{1}{\phi(x)} = \omega^2$$

$$\frac{EI}{\rho S}\frac{d^4\ \phi(x)}{dx^4} - \omega^2\phi(x) = 0$$

$$\frac{d^4\ \phi(x)}{dx^4} - \frac{\rho S}{EI}\omega^2\phi(x) = 0 \tag{10}$$

By setting

$$k^4 = \frac{\rho S}{EI}\omega^2 \tag{11}$$

We obtain

$$\frac{d^4\ \phi(x)}{dx^4} - k^4\phi(x) = 0 \tag{12}$$

The characteristic equation associated with this differential equation is

$$r^4 - k^4 = 0 \tag{13}$$

$$r^4 = k^4$$

This gives

$$r_1 = k \, ; r_2 = -k \, ; r_3 = jk \, ; \, r_4 = -jk \qquad (14)$$

The solution of the equation (12) can be put in the form

$$\phi(x) = a\sin(kx) + b\cos(kx) + c\,\mathrm{sh}(kx) + d\,\mathrm{ch}(kx) \qquad (15)$$

According to the relation (11)

$$k^4 = \frac{\rho S}{EI}\omega^2$$

$$k^2 = \sqrt{\frac{\rho S}{EI}\omega^2} = \sqrt{\frac{\rho S}{EI}}\,\omega$$

Then

$$\omega = k^2\sqrt{\frac{EI}{\rho S}} \qquad (16)$$

But

$$\omega = 2\pi f \qquad (17)$$

So the natural frequency of the beam is

$$f = \frac{k^2}{2\pi}\sqrt{\frac{EI}{\rho S}} \qquad (18)$$

The natural pulsations of the beam subjected to a benging vibration for values k_n of k are obtained by

$$\omega_n = k_n^{\,2}\sqrt{\frac{EI}{\rho S}} \qquad (19)$$

The natural frequency f_n corresponding to ω_n have as expression

$$f_n = \frac{k_n^{\,2}}{2\pi}\sqrt{\frac{EI}{\rho S}} \qquad (20)$$

The natural mode corresponding to f_n are represented by

$$\phi_n(x) = a\sin(k_n x) + b\cos(k_n x) + c\,\mathrm{sh}(k_n x) + d\,\mathrm{ch}(k_n x) \qquad (21)$$

The constants a, b, c and d are determined by the application of the boundary conditions depending on the type of connection of the ends of the beam.

3. APPLICATIONS TO THE TUBES

3.1 Tube characteristics

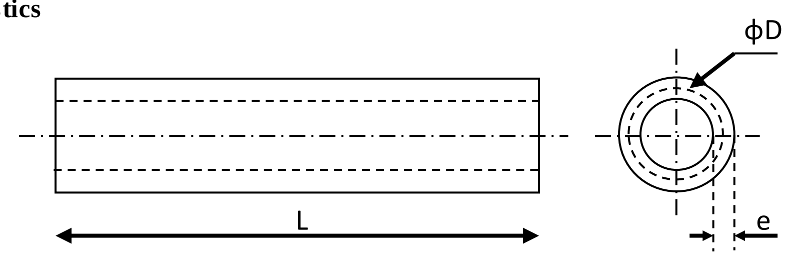

Fig-3 : Représentation of a tube

Table-1 : Some characteristics of the tubes to be studied

Type of material	**Longitudinal modulus of elasticity E** [MPa]	**Volumic mass ρ** [kg/m³]	**Average diameter D** [mm]	**Thickness e** [mm]	**Length L** [m]
Composite Carbone/Epoxy	120000	1500	60	1	1
Steel	210000	7800	60	1	1

The cross-sectional area of the tube is

$$S = \frac{\pi}{4}\left[(D+\frac{e}{2})^2 - (D-\frac{e}{2})^2\right]$$

The quadratic moment of the section is (22)

$$I = \frac{\pi}{64}\left[(D+\frac{e}{2})^4 - (D-\frac{e}{2})^4\right]$$

3.2 Case of tube whose two ends are supported

Fig-4 : Representation of the tube with two supported ends

The constants a, b, c and d of the relation (21) can be determined by writing the boundary conditions

$$\left(\phi_n(x)\right)_{x=0} = 0 \qquad (24)$$

$$\left(\frac{d^2\phi_n(x)}{dx^2}\right)_{x=0} = 0 \qquad (25)$$

$$\left(\phi_n(x)\right)_{x=L} = 0 \qquad (26)$$

$$\left(\frac{d^2\phi_n(x)}{dx^2}\right)_{x=L} = 0 \qquad (27)$$

Conditions (24) and (25) lead to

$$b = d = 0$$

$$\phi_n(x) = a\sin(k_n L) + c\,\mathrm{sh}(k_n L) \qquad (28)$$

Conditions (26) and (27) give

$$a\sin(k_n L) + c\,\mathrm{sh}(k_n L) = 0 \qquad (29)$$

$$-a\sin(k_n L) + c\,\mathrm{sh}(k_n L) = 0 \qquad (30)$$

Discarding the case where a and c are simultaneously zero, the next determinant must be zero

$$\begin{vmatrix} sin(k_n L) & sh(k_n L) \\ -sin(k_n L) & sh(k_n L) \end{vmatrix} = 0 \quad (31)$$

By developing, we get

$$\sin(k_n L).\,\mathrm{sh}(k_n L) = 0 \quad (32)$$

This implies

$$\sin(k_n L) = 0$$

Then $k_n L = n\pi$

$$k_n = \frac{n\pi}{L} \quad (33)$$

The relation (19) allows us to write

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}} \quad (34)$$

But

$$f_n = \frac{\omega_n}{2\pi}$$

$$f_n = \frac{n^2\pi}{2L^2}\sqrt{\frac{EI}{\rho S}} \quad (35)$$

Relations (29) and (32) give

$$c = 0$$

Relation (28) becomes

$$\phi_n(x) = a\sin\left(\frac{n\pi}{L}L\right)$$

By choosing a=1, the natural modes represented by the function $g_n(x)$ for the tube with supported ends has for expression

$$g_n(x) = \sin\left(\frac{n\pi}{L}L\right) \quad (36)$$

3.3 Case of tube whose two ends are fixed

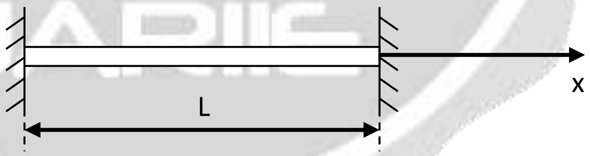

Fig-5 : Representation of the tube with two fixed ends

To determine the constants a, b, c and d of the function $g_n(x)$, we write the boundary conditions

$$\left(\phi_n(x)\right)_{x=0} = 0$$

$$\left(\frac{d\phi_n(x)}{dx}\right)_{x=0} = 0$$

$$\left(\phi_n(x)\right)_{x=L} = 0$$

$$\left(\frac{d\phi_n(x)}{dx}\right)_{x=L} = 0$$

These four conditions lead to

$$b + d = 0$$

$$a + c = 0$$

$$a\sin(k_nL) + b\cos(k_nL) + c\,\mathrm{sh}(k_nL) + d\,\mathrm{ch}(k_nL) = 0$$

$$a\cos(k_nL) - b\sin(k_nL) + c\,\mathrm{ch}(k_nL) + d\,\mathrm{sh}(k_nL) = 0$$

Solutions other than identically zero solutions imply that the determinant of this system is zero

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ sin(k_nL) & cos(k_nL) & sh(k_nL) & ch(k_nL) \\ cos(k_nL) & -sin(k_nL) & ch(k_nL) & sh(k_nL) \end{vmatrix} = 0$$

The calculations give

$$cos(k_nL).ch(k_nL) = 1 \quad (37)$$

$$cos(k_nL) = \frac{1}{ch(k_nL)} \quad (38)$$

We search

$$\alpha_n = k_nL \quad (39)$$

The values of α_n correspond to the points P_n which are the intersections of the curves representing the functions $cos(k_nL)$ et $\frac{1}{ch(k_nL)}$.

The natural frequencies f_n of the beam for the values k_n of k are obtained by relation (20).

$$k_n = \frac{\alpha_n}{L}$$

$$f_n = \frac{{\alpha_n}^2}{2\pi L^2}\sqrt{\frac{EI}{\rho S}} \quad (40)$$

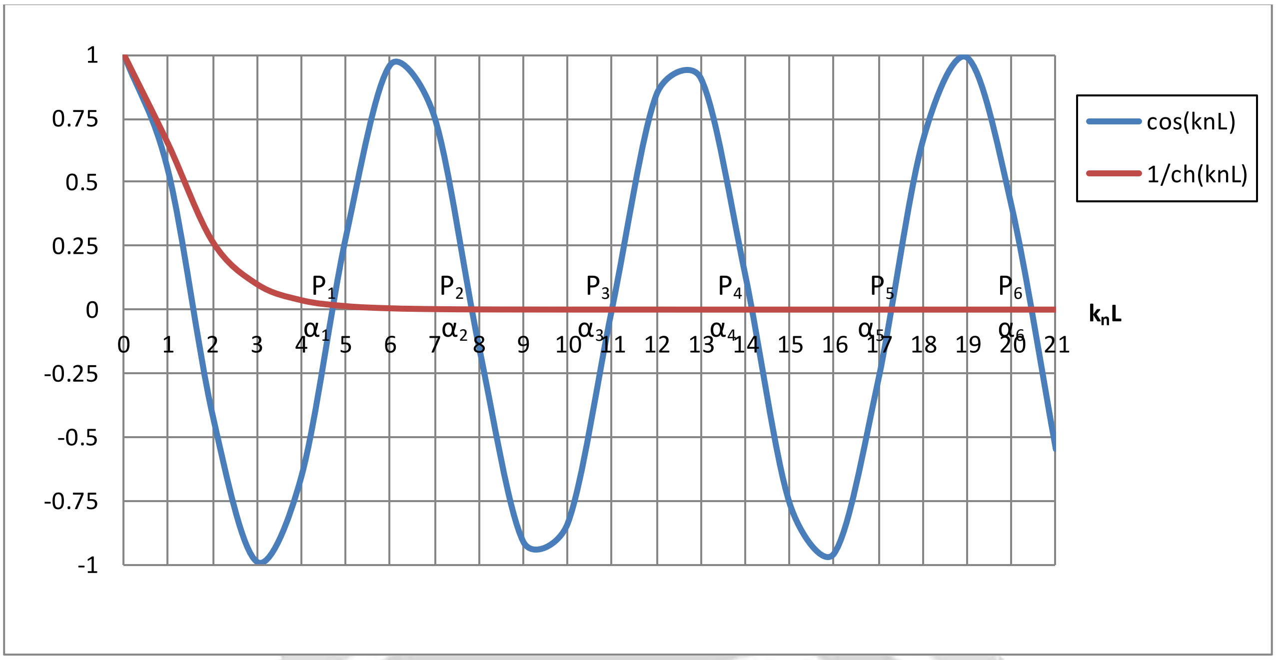

Chart-1 : Representation of the $\propto_n$ of the tube with fixed ends

The first six values of $\propto_n$ are shown in the table below.

Table-2 : Some values of $\propto_n$ for the tube with fixed ends

n	1	2	3	4	5	6
$\propto_n$	4,73	7,85	11	14,14	17,28	20,42

By choosing a=1, we get

$$c = -1$$

$$b = \frac{\mathrm{sh}(k_n L) - \sin(k_n L)}{\cos(k_n L) - \mathrm{ch}(k_n L)}$$

$$d = -b$$

The natural modes of the relation (21) have as expression

$$g_n(x) = \sin(k_n x) - \mathrm{sh}(k_n x) + \frac{(\mathrm{sh}(k_n L) - sin(k_n L))}{(\cos(k_n L) - ch(k_n L))}(\cos(k_n x) - \mathrm{ch}(k_n x)) \quad (41)$$

3.4 Case of tube whose one end fixed and the other free

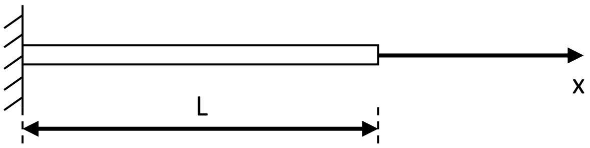

Fig-6 : Representation of the tube with one end fixed and the other free

The boundary conditions are

$$\left(\phi_n(x)\right)_{x=0} = 0$$

$$\left(\frac{d\phi_n(x)}{dx}\right)_{x=0} = 0$$

$$\left(\frac{d^2\phi_n(x)}{dx^2}\right)_{x=L} = 0$$

$$\left(\frac{d^3\phi_n(x)}{dx^3}\right)_{x=L} = 0$$

These conditions lead to the following system of four equations

$$b + d = 0$$

$$a + c = 0$$

$$-a\sin(k_nL) - b\cos(k_nL) + c\,\mathrm{sh}(k_nL) + d\,\mathrm{ch}(k_nL) = 0$$

$$-a\cos(k_nL) + b\sin(k_nL) + c\,\mathrm{ch}(k_nL) + d\,\mathrm{sh}(k_nL) = 0$$

Note that the determination of the constants a, b, c and d is only possible if the determinant of this system of equations is zero

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -sin(k_nL) & -cos(k_nL) & sh(k_nL) & ch(k_nL) \\ -cos(k_nL) & sin(k_nL) & ch(k_nL) & sh(k_nL) \end{vmatrix} = 0$$

After developping, we get

$$cos(k_nL).ch(k_nL) = -1 \quad (41)$$

$$cos(k_nL) = -\frac{1}{ch(k_nL)} \quad (42)$$

We determine

$$\alpha_n = k_nL$$

The points of intersection of the curves representing the functions $cos(k_nL)$ et $\left(-\frac{1}{ch(k_nL)}\right)$ give the value of α_n.

The natural frequencies f_n have for expressions

$$f_n = \frac{{\alpha_n}^2}{2\pi L^2}\sqrt{\frac{EI}{\rho S}} \quad (43)$$

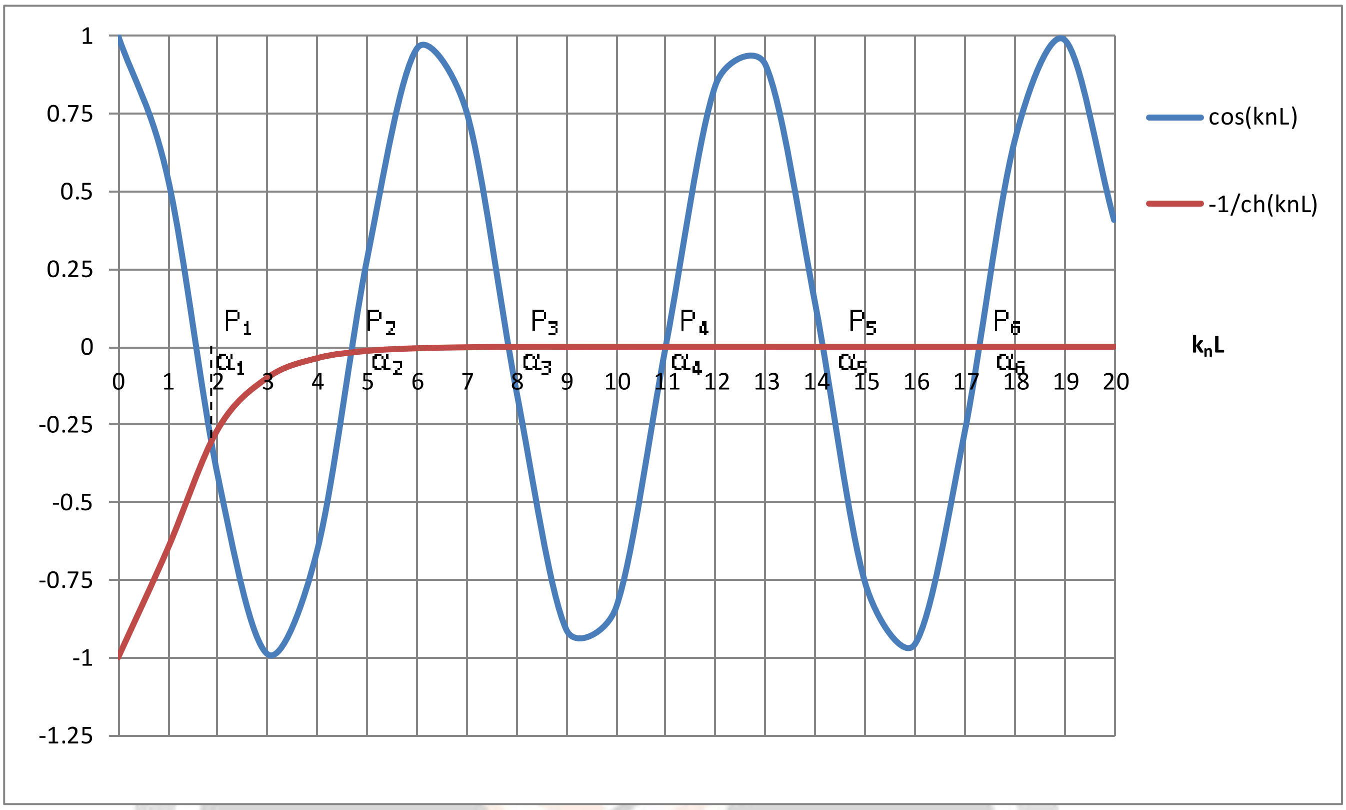

Chart-2 : Representation of the $\propto_n$ of the tube with one fixed end and the other free

Table-3 : Some values of $\propto_n$ for the tube with one fixed end and the other free

n	1	2	3	4	5	6
$\propto_n$	1,88	4,69	7,85	10,99	14,14	17,28

By choosing a=1, we have

$$c = -1$$

$$b = \frac{\sin(k_n L) + \mathrm{sh}(k_n L)}{\cos(k_n L) + \mathrm{ch}(k_n L)}$$

$$d = -b$$

According to the relation (21), the natural modes are represented by $g_n(x)$

$$g_n(x) = \sin(k_n x) - \mathrm{sh}(k_n x) + \frac{(\sin(k_n L) + \mathrm{sh}(k_n L))}{(\cos(k_n L) + ch(k_n L))}(\cos(k_n x) - \mathrm{ch}(k_n x)) \quad (44)$$

3.5 Study results

The first six natural frequencies for the most common boundary conditions are represented by the charts 3 and 4. Similarly, the representative curves of the natural modes which correspond to them are drawn.

3.5.1 Natural frequencies of the studied tubes

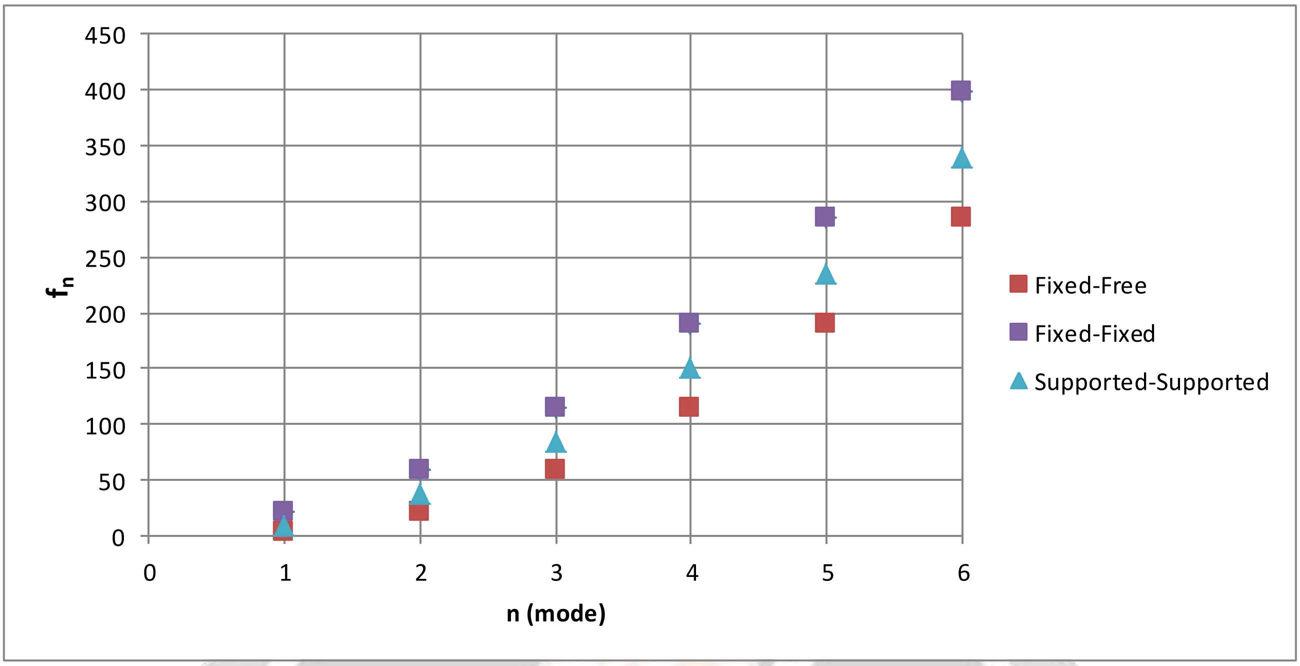

Chart-3 : Natural frequencies of the Carbone/Epoxy tube

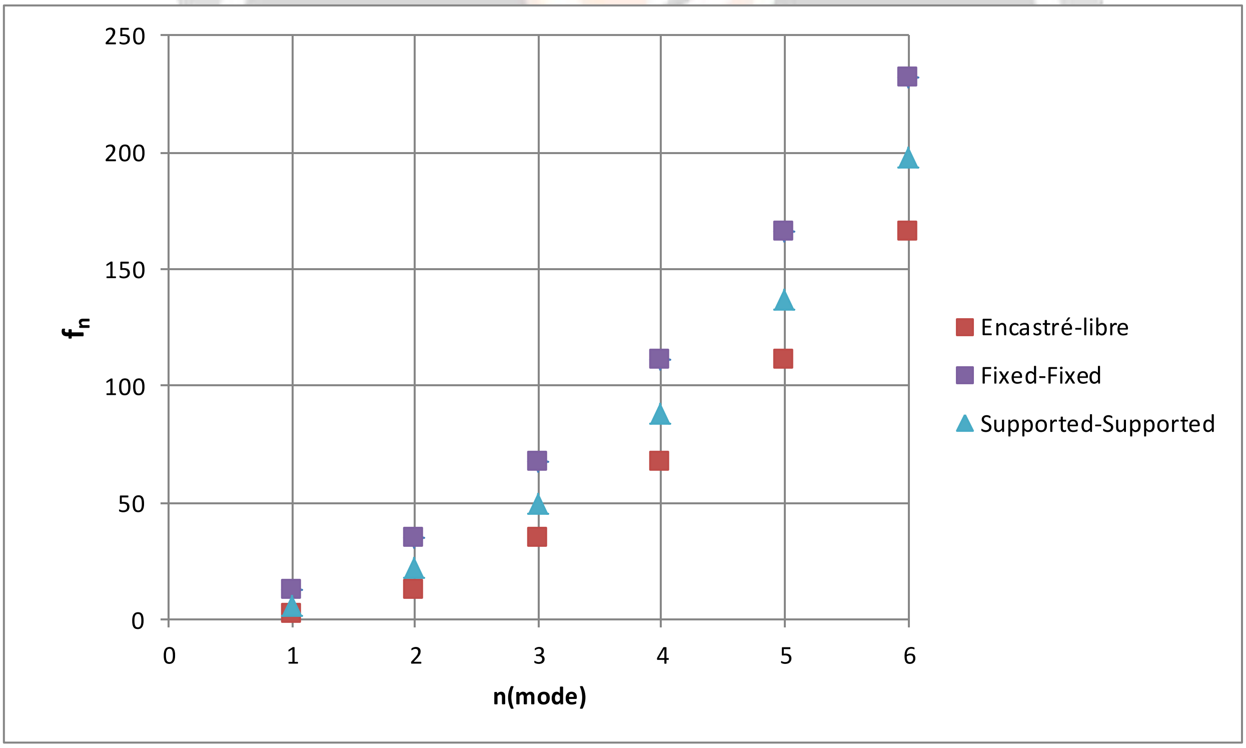

Chart-4 : Natural frequencies of the steel tube

Charts 3 and 4 show that the natural frequencies of the tubes studied in bending vibration increase with n for the three types of connection of the ends considered. These frequencies are higher when the ends are fixed.
It can also be noted that the natural frequencies of the tubes with supported ends represent approximately the average of those of the tube with fixed ends and of the tube with one fixed end and the other free. In all cases, in bending vibration, the Carbon/Epoxy tube has natural frequency values higher than those of the steel tube.

3.5.2 Natural modes of the studied tubes

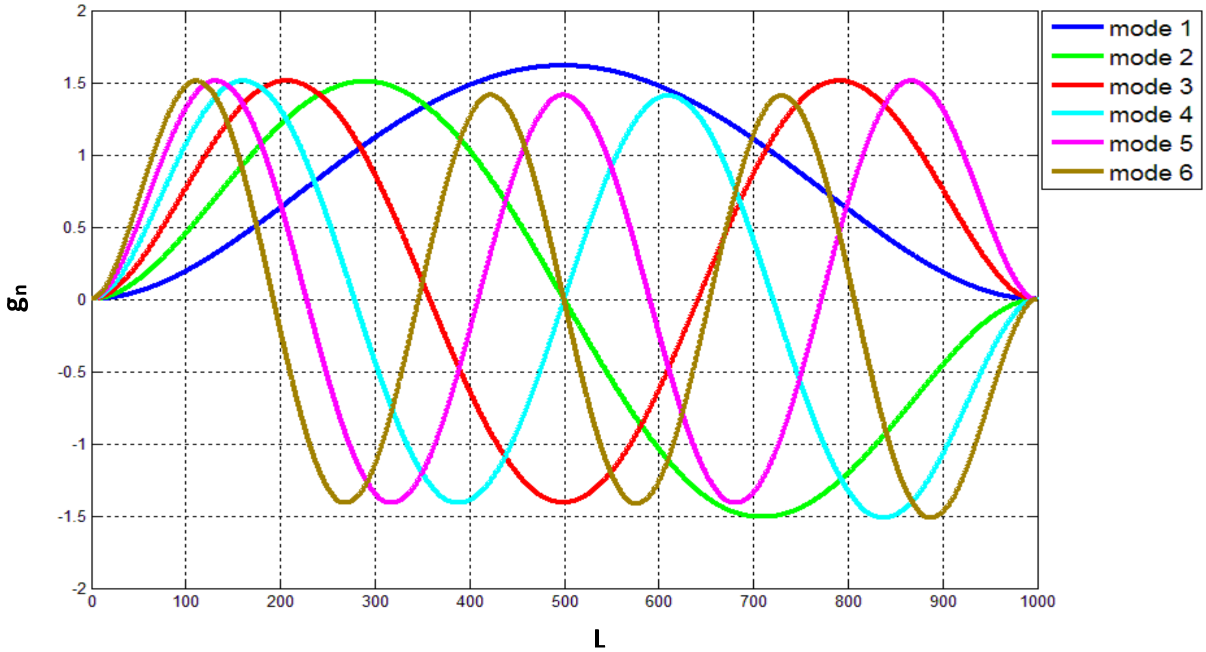

Chart-5 : Natural modes of the tube with fixed ends

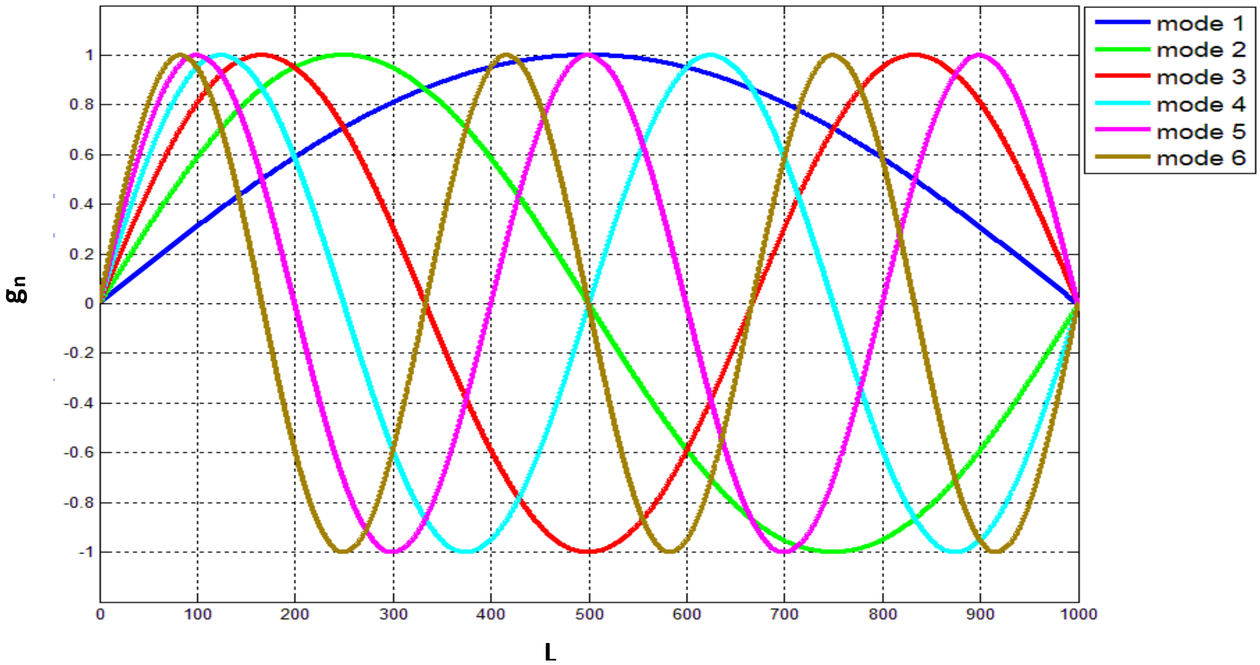

Chart-6 : Natural modes of the tube with supported ends

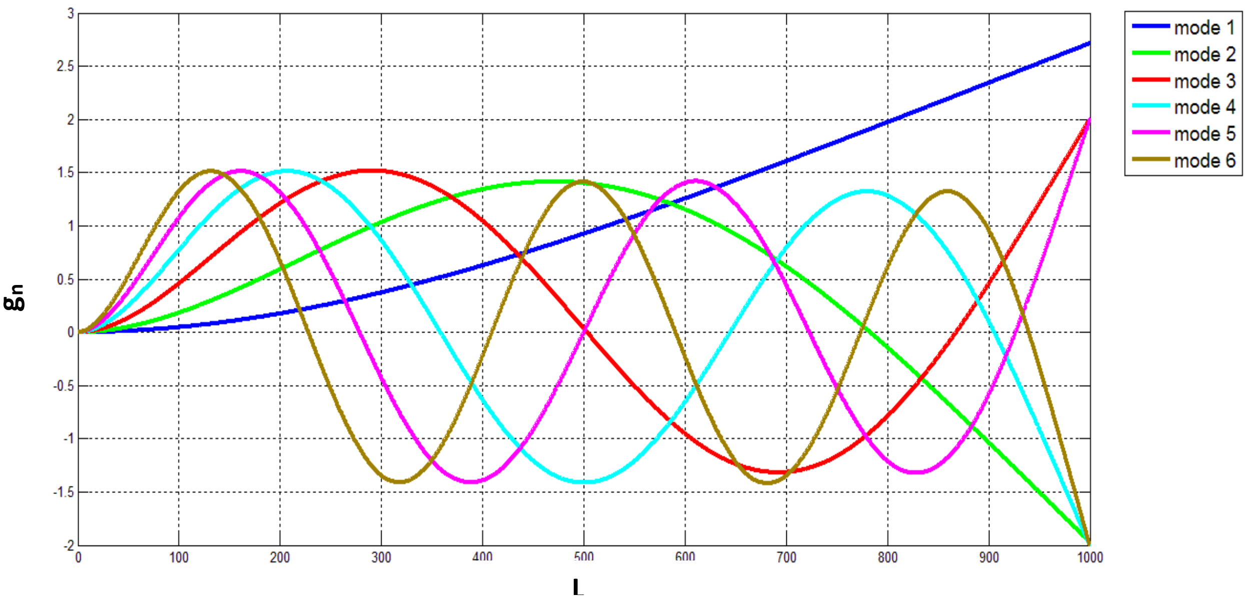

Chart-7 : Natural modes with one fixed end and the other free

The natural modes do not depend on the nature of the material but the charts 5, 6 and 7 show that they vary with the type of connection of the ends of the tube studied.
It is also noticed that except the case of the first mode (mode 1), the representative curves of the other natural modes are sinusoidal.

4. CONCLUSION

The purpose of this research work is to study the vibratory behavior in bending of two unloaded beams which are two tubes, one in Carbon/Epoxy composite and the other in steel, of constant circular section and of the same dimensions and non-negligible masses.
To avoid the phenomenon of amplitude resonance which is generally harmful for an element constituting an industrial machine, it is important to know how to determine the natural frequencies and natural modes of this element.
The results of this work carried out on the two tubes showed that the nature of the material and the type of connection of the ends of the beam have an influence on the natural frequencies of this beam in bending vibration.
Indeed, the Carbon/Epoxy composite tube has higher natural frequency values than those of the steel tube. These values are even higher when the two ends are fixed. It is also noted that these natural frequencies increase with the order of the mode.
Concerning the natural modes, they depend only on the type of connection of the ends and their representative curves make it possible to note that all these curves start from the same point for a type of connection considered.

5. REFERENCES

[1] Babakov I. M., « *The theory of oscillation* » Nauka, Moscow, 1968

[2] Courbon J., « *Vibrations des poutres* ». Annales de l'Institut du Bâtiment et des Travaux Publics, 1969

[3] M. Geradin et D. Rixen, « *Théorie des vibrations* », Masson, 1993

[4] W. Timoshenko, « *Théorie des vibrations* », Bellenger, 1947

[5] Tse F. S., Morse I.E., Hinkle R. T., « *Mechanical vibrations*», Allyn and Bacon Inc., Boston, 1963