Some Refinement of Polya and Heinz Operator Inequalities

Author: Mohammad Zarif Mehrzad,

lecturer of Mathematics department, parwan University

Abstract

In this paper, first we briefly reviewed some of the important concepts of mathematics, including Banach space, Hilbert space, numerical mean, and mean operator. Then there are several improvements in Heinz and Polya inequalities for real numbers and operators. And, at the end of the use of Heinz and Ponia inequalities, we study the classical Polius inequality for real numbers and operators in the Hilbert space.

Keywords: *Banach Space, Hilbert Space, Polya Inequality, Operator Average*.

Introduction

The theory of operators in Hilbert spaces is an important branch of functional analysis and operator theory, which, by utilizing the multiplicative property of Hilbert spaces, in particular the Cauchy-Schwartz inequality, and the Rees show theorem for the Hilbert space duality, yields results in mathematical analysis, which reach to an applicable results in mathematics analysis [1-3]. Its application can be seen in statistics, numerical calculations, numerical analysis [4], and engineering. One of the classical inequalities in mathematics is Heinz and Polia's inequality [5], which examines the mean of Haynes and its relation to arithmetic, geometric, logarithmic, and hronometric meanings [6]. This inequality has been improved in various ways and its application has been obtained in the operator theory of Hilbert space [7-9].

In this paper, the inequalities of Heinz-Polya and its application in operators is studied.

Banach space and space C-algebra

Definition 1.2.1. A vector space A on field C is an algebra. If a multiplicity is defined on it, the mapping xy (x; y) from A × A to A, which applies to any x, y, z A, and any c in the following properties:

$$(1) x(yz) = (xy)z$$
$$(2) x(y+z) = xy + xz$$
$$(3) (x+y)z = xz + yz$$
$$(4) (ax)y = a(xy) = x(ay)$$

The algebra A is named shift for any $x, y, A: xy = yx$.

Definition 1.2.2. The vector subfield S of an algebra A is called A subalgebra A. if for any x, y $\in$ S, x, y $\in$ S. It is obvious that a sub-algebra S of A, itself with a C field and a multiplication of heritable algebra of A, is an algebra.

Definition 1.2.3. Assume that X is a vector space on C. A seminorm on X is a mapping of P from X to R such that for each y $\in$ X and each $\alpha \in F$.

$$1 P(ax) = |a| P(x)$$
$$2 P(x+y) \leq P(x) + P(y)$$

A norm on X is a semi norm P of this property if p (x) = 0. Then x = 0, usually norm is shown with the symbol $\|\cdot\|$ X along with $\|\cdot\|$ is named semi norm space.

Definition 1.2.4. The normed vector space X on the field C is a Banach space. If X along with norm $\|\cdot\|$ is completed, That is, every Cauchy sequence in norm $\|\cdot\|$ convergence.

Hilbert spaces

Definition 1.3.1. The complex vector space H is an inner space whenever the two vectors x and y in H are assigned to the complex number (y, x) such that for each $x, y, z \in H$, and $\alpha \in c$ have

$$1 (x+y, z) = (x, z) + (y, z)$$
$$2 (x, y) = \overline{(y, z)}$$
$$3 (ax, y) = a(x, y)$$
$$4 (x, x) \geq 0$$
$$5 (x, x) = 0 \Leftrightarrow x = 0$$

If $(0,0)$ is inner product in on vector space H, Couple $(H, \langle x, y \rangle)$ is named a complex inner multiplication space and a complex number (x, y) for every x, y, H the inner multiplication of x and y.

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$\langle x, \alpha y \rangle = -\alpha \langle x, y \rangle$$

Definition 1-3-5: In the inner space of H, norm X with the symbol $\|.\|$ is defined as below:

$\|x\|^2 = \langle x, x \rangle$

Using the above definition, Cauchy-Schwartz inequality in inner multiplication space H can be written as follows:

$|\langle x, x \rangle| \leq \|x\|.\|y\|$

Theorem 1.3.1. If x and y are in the inner multiplication space H, then $\|x, y\| \leq \|x\|.\|y\|$

Definition 1.3.2. If $(0,0)$ is an inner multiplication on the complex vector space H, then with the definition of $\|x, y\| = (x, x)^{\frac{1}{2}}$ for each $x \in H$, H is a norm-squared linear space, which this norm is produces norm by inner multiplication. If inner space of H by $\|x, y\| = (x, x)^{\frac{1}{2}}$ norm is completed, that is Hilbert space.

3. The expansion of Heinz and Polya inequality

Suppose that, $0 \leq u \leq 1, ab \geq 0$, $\varphi(u) = H_u(a,b)$, φ relative to $u = \frac{1}{2}$, symmetric

In other words,

$$\varphi(u) = u(1-u), u \in [0,1] \text{ for } i = 0,1,\ldots,n-1, n = 1,2,\ldots :$$

$$E_{n,i} = \left[2^{-n} i, 2^{-n}(i+1) \cup (1-2^{-n}(i+1), 1-2^{-n} i)\right]$$

Then defined:

$$\varphi_n(u) \sum_{i=0}^{2^{n-1}-1} \left[\left(2^n T \cdot -i\right)\varphi\left(2^{-n}(i+1)\right) + (i+1-2^n T \cdot)\varphi(2^{-n} i)\right] X E_{n,i,\ldots}(u)$$

$$\varphi_n\left(\frac{1}{2}\right) = \sqrt{ab}, \mathrm{T}_{\square} = \min\{u, 1, -u\}, u \neq \frac{1}{2}$$

Lemma 3.1. Assume that ϕ (v) = Hv (a: b) is convex, then:

$$\varphi(u) \leq (2^n t_{.} - i)\varphi(2^{-n}(i+1)) + (i+1-2^n t)\varphi(2^{-n} i)$$

Proof: With the assumption $v \leq \frac{1}{2}$ and

$$= 2^n v\,\varphi(2^{-n}(i+1)) - iv\,(2^{-n}(i+1)) + (i+1)\varphi(2^{-n}i) - 2^n v\,\varphi(2^{-n}i)$$
$$2^n\left[\varphi(2^{-n}(i+1)) - \varphi(2^{-n}i)\right]v - \left[i\,\varphi(2^{-n}(i+1))2^{-n}(i+1)\varphi(2^{-n}i)\right]$$
$$= \frac{\varphi(2^{-n}(i+1)) - \varphi(2^{-n}i)}{2^{-n}}v - \frac{(2^{-n}i\,\varphi(i+1))2^{-n}(i+1)\varphi(2^{-n}i)}{2^{-n}}$$

According to the hypothesis 1.1.2. conclude that:

$$\varphi(v) \le \left(2^n r_0 - i\right)\varphi(2^{-n}(i+1)) + (i+1-2^n r_0)\varphi(2^{-n}i)$$

For $v \le \frac{1}{2}$

Lemma 3.2 $\varphi(v) = H_v(a,b)$ Then $\int_0^1 \varphi(v)dv = \int_0^1 a^v b^{1-v} dv = L(a,b)H_v(a,b)$

Proof:

$$\int_0^1 \varphi_n(v) = \int_0^1 \frac{a^v b^{1-v} + a^{1-v} b^v}{2} dv = \frac{b}{2}\int_0^1 \left(\frac{a}{b}\right)^v dv + \frac{a}{2}\int_0^1 \left(\frac{b}{a}\right)^v dv$$
$$= \frac{a-b}{2\log\frac{a}{b}} + \frac{b-a}{2\log\frac{b}{a}} = \frac{a-b}{\log a - \log b}$$
$$= \int_0^1 a^v b^{1-v} dv$$
$$= L(a,b)$$

Lemma 3.3: Assuming 4.2 relationships is obtained,

$$\int_0^1 \varphi_n(v) = 2^{-n}\sum_{i=0}^{2^{n-1}-1}\left[\varphi(2^{-n}(i+1)) + \varphi(2^{-n}i)\right]$$
$$= 2^{-n}\left[\sqrt{ab} + \frac{a+b}{2} + 2\sum_{i=0}^{2^{n-1}-1}\varphi(2^{-n}i)\right]$$

Proof: Consider the diagram $\varphi_n(v)$ in period $\left[\frac{i}{2^n}, \frac{i+2}{n}\right]$ of and $\varphi_n(v)$ symmetric.

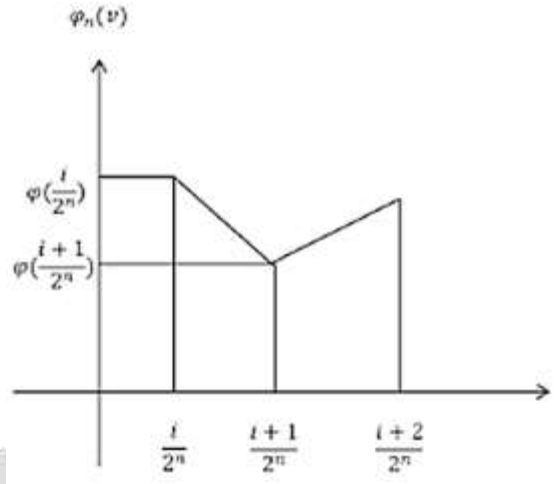

Figure 3:

To get the left trapezoidal area

$$\frac{1}{2^n}.\frac{\varphi(\frac{i}{2^n})+\varphi(\frac{i+1}{2^n})}{2}$$

Due to the symmetry property, we double this area

$$\frac{1}{2^n}(\varphi((2^{-n}i)+\varphi(2^{-n}(i+1)))$$

Now for $0 \le v \le 1$ and $i = 0,1,,...,2^n - 1$ get:

$$= \int_0^1 a^v b^{1-v} dv = 2^{-n} \sum_{i=0}^{2^{n-1}-1} \left[\varphi(2^{-n}(i+1)) + \varphi(2^{-n}i) \right]$$

$$-2^{-n} \left[\sqrt{ab} + \frac{a+b}{2} + 2 \sum_{i=0}^{2^{n-1}-1} \varphi(2^{-n}i) \right]$$

4- Polya Inequality for Operators in Hilbert Space:

4.1. Functional calculus and operational form of Heinz and Heron inequalities

Assume that $B(H)$ is the set of all linear operators in a Hilbert space H, operators $A(H)$ are positive and write $A \ge 0$ and $A(x,x) \ge 0 \ge 0$. For each vector $x \in H$, If A and B self adjoint operators. The regularity of A means than $B - A$ is a positive operator.

Proposition 1.1.4. If $A \in B(H)$, then A spectrum as a symbol of $\sigma(A)$ is defined as:

$\sigma(A)=\{A\in C / A=I \; and \; B(H) \, is \, invertible\}$

According to the Banach theorem, $\sigma(A)$ non-empty and compact subset.

In order to obtain the inequality of linear arbitrary linear operators in Hilbert space, we use the following uniformity function for operator functions.

Lemma 4.1. Suppose that $A\in B(H)$ is a self-adjoint operator and g,h are the real continuous functions. When, $f(t)\geq g(t)$, for every $t\in\sigma(A)$, then $f(A)\geq g(A)$.

Proof: If $h=f-g$,then $h=kk, h=\sqrt{f-g}$,

$E_A(h) = E_A(kk) = E_A(k)E_A(k) = E_A(k)''.E_A(k)$

However, $B(H)$ is the operators is in the form of nonnegative $B*B$.

Then $E_A(h) = f(A)-g(A)$ and $E_A(h)\leq 0.$, $f(A)\geq g(A)$

4.2. Riemann integral expansion to Banach function values

It is supposed that E is a Banach function and $a,b\in R$,Then the set of all bounded functions $f:[a,b]\rightarrow R$ is shown with the symbol of $B([a,b],E)$.

Proposition 4.2.1. $B([a,b],E)$ with norm $EB([a,b],E)$ if $\|f\|_\infty = \sup_{a<x<b}\|f(x)\|$ is a Banach space.

Proof. It is clear that $B([a,b],E)$ is a norm space, it is enough to show that $B([a,b],E)$ is Banach space. If $\{f_n\}_{n=l}^{\infty}$ sequence $\|.\|_\infty$ Cauchy in $B([a,b],E)$ if $x\in[a,b]$ due to $\|f_n(x)-f_m(x)\|\leq\|f_n-f_m\|_\infty$, Then $\{f_n(x)\}_{n=l}^{\infty}$ is a Cauchy sequence in E. Hence, $e_x\in E$ and $f_n(x)\rightarrow e_x$.

We defined that $f(x) = e_x$ ، $f:[a,b]\rightarrow E$, Then show that f is a bounded function and For this work, $\varepsilon>0$ due to the $\{f_n\}$ is a Cauchy sequence, $n_\circ \in$ N, $(m,n\geq n_0)$ and $\|f_n(x)-f_m(x)\|_\infty\rightarrow\frac{\varepsilon}{2}$ Then:

$$\|f_n(x)-f_m(x)\|\leq\|f_n-f_m\|_\infty<\frac{\varepsilon}{2}, x\in[a,b], m,n\geq n_0.$$

In above equation if $m\rightarrow\infty$ due to $f_m(x)\rightarrow f(x)$, Then $\|f_n(x)-f(x)\|\leq\frac{\varepsilon}{2}, x\in[a,b],\; n\geq n_0$

Hence, $\|f_n-f\|_\infty\leq\frac{\varepsilon}{2}<\varepsilon, n\geq n_0$, So $\|f_n-f\|_\infty\rightarrow 0 \; and \; \|f_n-f\|_\infty<\varepsilon$ especially

$\|f\|_\infty\leq\|f_{n_0}-f\|_\infty+\|f_{n_0}\|_\infty<\varepsilon+\|f_{n_0}\|_\infty<\infty$ Then $f\in B([a,b],E) \; and \; f_n \|.\|_\infty \; f$

4.3. Improving Heinz inequality in operating mode

Theorem 4.3.1. Suppose that g:(0, ∞) × [0,1] $\longrightarrow$ R is continuous function and $f_n(x) = \frac{1}{n}\sum_{i=1}^{n} g(x, \frac{1}{n})$, n∈N,

Then the sequence $\{f_n\}$ on every closed interval [m,M] to $f(x) = \int_0^1 g(x,t)dt$ function is a uniform convergent.

Proof: due to g in [m,M] × [0,1] interval is continious and [m,M] × [0,1] interval is compacted, then g for continious interval is uniform. If ϵ >0, then δ>0, if *x*, y ∈ [m,M] × [0,1] and $|x - y| < \delta$ and $|f(x)-f(y)| < \epsilon$. So, $n_\circ \in N$ and $1/n< \delta$. Then for $n \geq n_\circ$, $1/n< \delta$. Hence, for every x ∈ [m,M], if $|t_2-t_1|<1/n$. So $|g(x, t_1)- g(x, t_2)| < \epsilon$. In according to this point, if $n \geq n_\circ$, then for every x ∈ [m,M]

$$|f_n(x) - f(x)| = |\frac{1}{n}\sum_{i=1}^{n} g(x, \frac{i}{n}) - \int_0^1 g(x,t)dt|$$

$$= |\frac{1}{n}\sum_{i=1}^{n} g(x, \frac{i}{n}) - \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} g(x,t)dt| = |\sum_{i=1}^{n}(\frac{1}{n}g(x,\frac{i}{n})) - \int_{\frac{i-1}{n}}^{\frac{i}{n}} g(x,t)dt|$$

$$\leq \sum_{i=1}^{n} |\frac{1}{n}g(x,\frac{i}{n}) - \int_{\frac{i-1}{n}}^{\frac{i}{n}} g(x,t)dt| = \sum_{i=1}^{n} |\int_{\frac{i-1}{n}}^{\frac{i}{n}} g(x,\frac{i}{n})dt - \int_{\frac{i-1}{n}}^{\frac{i}{n}} g(x,t)dt|$$

$$\leq \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} |g(x,\frac{i}{n}) - g(x,t)|dt \leq \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \varepsilon dt = \varepsilon$$

Then

$$\sup_{x\in[m,M]} |f_n(x) - f(x)| \leq \varepsilon$$

So, f_n is uniform convergent on f.

Theorem 4.2.1. if $g:(0,\infty)\times[0,1] \to R$ is continuous and $f(x) = \int g(x,t)dt, x \in (0,\infty)$ If A is a positive operator on the Hilbert space H, In other words $A = A^*$ $\sigma(A) \subseteq (0,\infty)$, Then $f(x) = \int g(A,t)dt$

Proof. *If x* ∈ (0,*1*) and $f_n(x) = \frac{1}{n}\sum_{i=1}^{n} g(x, \frac{i}{n})$, due to the σ (A) is compact and $\sigma(A) \subseteq (0,\infty)$ and, $m, M > 0, \sigma(A) \subseteq [m.M]$.

In according to the Theorem 4.3.1. f_n to f on [m,M] and σ (A) is uniform convergent and in according to the Functional calculus

$$f(A) = \lim f_n(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g(A, \frac{1}{n})$$

But, $h:[m,M] \to B(H), t \to g(A,t)$ is continuous. So, h ∈ Reg ([m,M],B(A)), then

$$\int_0^1 h(t)dt = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h(\frac{i-1}{n})$$

In other words:

$$\int_0^1 g(A,t)dt = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(t, \frac{i}{n})$$

So, $f(A) = \int_0^1 g(A,t)dt$.

Conclusion

In this study, some number of refinements of the Heinz inequality for real numbers and operators are presented. By using them, some refinements of the classical Pólya inequality and their operator versions are resulted.

References

[1] Shafiei M. and Ghazanfari A.G. Numerous refinement of polya and heinz operator inequalities, Linear and multilinear algebra, 2017.

[2] Bathia R., Interpolating the arithmetic – geometric mean inequality and its operator version linear Algebra appl. 2006: 413-355-363.

[3] Bhatia, R., Matrix analysis, Springer-Varlag, New York, 1997.

[4] Bhatia, R., The logarithmic mean, Resonance, 583-594, 2008. 25, 28.

[5] Bonsall, F.F., Duncan, J., Complete normed algebra, Springer- Varlag, New York, 1973.

[6] Furuchi S., Yanagi K. Bounds of the Logarithmic mean. J. Iequal Appl. 2013,

Articlle 535.

[7] Zou, L., Jiang, Y., Improved Heinz inequality and its application. J Inequal Appl. 2012. Article ID: 113.

[8] Zou, L., Matrix versions of the classical Polya inequality. Sciences Asia. 2013. 39: 204-207.

[9] Kittaneh F., Manarsh Y., Improved Young and Heinz inequalities for matrices. J. Math Anal Appl. 2010; 36: 262-269.