TO SOLVE DIFFERENTIAL EQUATION USING POLYNOMIAL INTEGRAL TRANSFORM

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ABSTRACT

In this paper, we discussed about polynomial integral transform for solving differential equations and also we use Laplace transform polynomial integral transform solving differential equations with a little effort. Here the integral transform entails the function as its kernel. The Cauchy-Euler method transforms a linear differential equation into an algebraic equation and also we have discussed about the characterization of polynomial integral transform and its properties by using the differential equations.

Keyword : *Polynomial integral transform, Polynomial function, Kernel, Differential equations.*

1. INTRODUCTION

There are many approaches to search for solution to differential equation with variable coefficients. The Cauchy-Euler method transforms a linear differential equation into an algebraic equation with the use of appropriate substitution technique. In addition these classical methods for search of solutions to the differential equations are tedious and cumber-some as one has to look for the appropriate substitution expression. Thus, there is no single substitution expression for a single type of differential equation. Nowadays integral transform method is the concern of mathematicians and scientists in general. Since the introduction of the Laplace integral transform, have been proposed for solving differential equations. An alternative integral transform, laplace substitution, for the construction of solutions of the partial differential equation was observed. We follows the outline of the paper. In section 1, we give the introduction to integral transform method. In this section we discuss the integral transform methods for solving differential equations. In section 1,2, we present the definition and also give the proof of the polynomial integral transform. Using the polynomial integral transform, we show that the solution of the differential equation converges for In section 3, we discuss about the properties of the polynomial integral transform In section 4, we apply the polynomial integral transform to derivatives, some ordinary differential equation and partial differential equation.   In section 5, it contains the conclusion of the paper.

1.1 Definition

An Integral Transform is any transform T of the following form

$$(Tg)(u) = \int_{k1}^{k2} k(\, t,\, u\,)\, g(t)\, dt$$

1.2 Definition

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called differential equation.

1.3 Definition
A polynomial of degree n is a function of the form
$$g(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$$

1.4 Definition

Let f (x) be a function defined for X > 1.Then the integral

$B(g(x)) = G (s) = \int_1^\infty g(1nx).\ x^{-s-1}\ dx,$

is the polynomial integral transform of g(x) for x $\in$ [1, ∞]

1.5 Definition

For any function g:A$\rightarrow$ B (where A and B are any sets), the kernel(also called the null space) is defined by
Ker (g) = x:x $\in$ Asuchthatg (x) = 0

1.6 Definition

A linear differential equation of the form

$$a_0 x^n\ (d^n y/dx^n) +\ a_1 x^{n-1}\ (d^{n-1}y/dx^{n-1}) + \ldots + a_{n-1}\left(\frac{dy}{dx}\right) + a_n y = x$$

Where $a_0 a_1 \ldots a_n$ are constants and X is either a constant or a function of x only is called a homogeneous linear differential equation. Note that the index of   and the order of derivative is same in each term of such equations. These are also known as Cauchy-Euler equations.

2. The polynomial integral transform

2.1 Theorem (A Polynomial integral transform)

Let g (x) be a function defined for x $\geq$ 1,Then the integral
$\qquad$ B (g(x)) = G (x) = $\int_1^\infty g\ (lnx).\ x^{-s-1}$ dx
is the polynomial integral transform of g (x) for x $\in$ [1, $\in$] , provided the integral converges.

Proof
We consider the homogeneous Cauchy-Euler equation of the form

$$a_n x^n\ (d^n y/dx^n) + a_{n-1}x^{n-1}\ (d^{n-1}y/dx^{n-1}) + \ldots + a_1 x\left(\frac{dy}{dx}\right) + a_0 y = 0$$

With the corresponding distinct roots

$Y(x) = c_1 e^{s1\ln x} +\ c_2 e^{s2\ln x} + \ldots + c_n e^{sn\ln x}$
Where $c_1\ c_2\ c_3$ are constants. Also , consider a constant linear differential equation
$a_n\ (d^n y/dt^n) + a_{n-1}\ (d^{n-1}y/dt^{n-1}) + \ldots + a_1\left(\frac{dy}{dt}\right) + a_0 y = 0$

With a solution
$c_2 e^{s2t} + c_n e^{sn\ t}$,

We can see that equation (3) has an integral transform

$\int_0^\infty g\ (t).\ e^{-st}$ dt

But Again , we see from equation

t = ln x

B $(g(x))$ = G $(x) = \int_1^\infty g\ (lnx).\frac{1}{x}\ e^{-sinx}\ dx = \int_1^\infty f\ (lnx)\ x^{-(s+1)}\ dx$
is the polynomial integral transform of g (x) for x $\in$ [1, ∞] provided the integral converges.

Hence the proof

2.1.1 The convergence of the polynomial integral transform

In this section we show that the polynomial integral transform converges for variable defined in [1, ∞]
By Taylor series expansion, we obtain

$e^{\ln x(-s-1)} = 1 + \ln x^{-s-1} + \ln x^{-(s+1)2}/\ 2! + \ln x^{-(s+1)2}/\ 3! +...+ \sum_{n=0}^\infty \ln x^{-(s+1)n}/\ n! + e^{\ln x(-s-1)} \sum_{n=0}^\infty \ln x^{-(s+1)n}/\ n!$

By the D'Lambert Ratio test, we obtain

$\lim_{n\to\infty} 1\ \frac{\sum_{n=0}^\infty \ln x^{-(s+1)n+1}}{(n+1)!} \div \frac{\ln x^{-(s+1)n}}{n!}$

$\to 0.\ln x^{(-s-1)} \to 0$

Then $B(g(x)) = sup_{1<x<\infty} \int_1^\infty Ig(\ln x).\ x^{-(s+1)}|\ dx$

$B(g(x)) = sup_{1<x<\infty} \int_1^\infty Ig(\ln x)||\ x^{-(s+1)}|\ dx$

$B(g(x)) \le M \int_1^\infty Ig(\ln x).dx$

Where M > 0

It implies that the polynomial integral transform converges uniformly for a given
The function g (x) must be piecewise continuous.
Thus, g (x) has at most a finite number of discontinuities on any interval $1 \le x \le A,$ and the limit of g(x) exist at every point of discontinuity,

2.1.2 Existence of the Polynomial Integral Transform

In this section, we show that the Polynomial Integral Transform exists for x $\in [1, \infty$].
 To see this, we state the existence theorem for the Polynomial Integral Transform.

2.2 Theorem

Let g(x) be a piecewise continuous function on $[1, \infty$] and of exponential
order, then the polynomial integral transform exists.

Proof
   By the definition of polynomial integral transform, we obtain

$I = \int_1^\infty g(x)\ x^{-(s+1)}\ dx$
  $= \int_1^A g(x)\ x^{-(s+1)}\ dx + \int_A^\infty g(x)\ x^{-(s+1)}\ dx$

So $I = I_1 + I_2$

Where $I_1 = \int_1^A g(x)\, x^{-(s+1)} dx$

$\quad\quad I_2 = \int_A^\infty g(x)\, x^{-(s+1)}\, dx$

The integral I_1 exists. Since $g(x)$ is piecewise continuous.

Taking $I_2 = \int_A^\infty g(x)\, x^{-(s+1)}\, dx$

$I_2 = \int_A^\infty g(x)\, x^{-(s+1)}\, dx \leq M \int_a^\infty e^{ax} \cdot x^{-(s+1)}\, dx$
By the Taylor series expansion, we obtain

$e^{ax} \approx \sum_{n=0}^\infty \infty^n\, x^n / n!$

Substituting the expression for e^{ax} in the above equation

We obtain
$I_2 = M\ \sum_{n=0}^\infty \frac{M\,\infty^n}{n!(s-n)}$, $s > n$

$\int_1^\infty g(x)\, x^{-(s+1)}\, dx = M \sum_{n=0}^\infty \frac{M\,\infty^n}{n!(s-n)}$, $s > n$

Hence the proof.

3.Properties of the Polynomial Integral Transform
In this section, we give the properties of the Polynomial Integral Transform

3.1 Theorem
The Polynomial Integral Transform is a linear operator.

Proof
Suppose that f(x) and g(x) are functions and $\propto_1, \propto_2$ and are real constants.

$B(\propto_1 f(x) + , \propto_2 g(x)) = \int_1^\infty (\propto_1\ f(\ln x)\ + , \propto_2\ g(\ln x)\ x^{-s-1}\, dx$
$B(\propto_1 f(x) + , \propto_2 g(x)\ = \propto_1 \int_1^\infty f(\ln x) \cdot x^{-s-1}\, dx + \propto_2 \int_1^\infty g(\ln x) \cdot x^{-s-1}\, dx$

So $B(\propto_1 f(x) + , \propto_2 g(x)) = \propto_1 B(f(x)) + \propto_2 B(g(x))$

Hence the proof.

3.2 Theorem

The Inverse Polynomial Integral Transform is also a linear operator

Proof
$B(\propto_1 f(x) + , \propto_2 g(x)) = \propto_1 B(f(x)) + \propto_2 B(g(x))$

Taking both sides the inverse integral transform to the above equation,
We obtain,

$\propto_1 f(x) + , \propto_2 g(x) = B^{-1}(\propto_1 B(f(x)) + \propto_2 B(g(x))$

α_1 f(x) + , α_2 g(x) = B^{-1} (α_1B ($f(x)$) + B^{-1} α_2B($g(x)$)
α_1 f(x) + , α_2 g(x) = α_1 B^{-1}(f(x)) + α_2 B^{-1}g(s))
α_1 f(x) + , α_2 g(x) = α_1 B^{-1}(f(x)) + α_2 B^{-1}g(s)))

Where B f(x)= f(s) and B(g (x) = G(s) respectively.
            Hence the proof.

3.3 Theorem
(First shifting theorem)
If $B(g(x))$ = B (s) then $B(e^{ax}g(x))$ = B(s-a), for s>1 .

Proof
Let $B(e^{ax}g(x)) = \int_1^\infty e^{alnx}$ g(ln x) x^{-s-1}dx
$B(e^{ax}g(x)) = \int_1^\infty x^a$ g(ln x) x^{-s-1}dx
$B(e^{ax}g(x)) = \int_1^\infty g(\ln x)$ $x^{-(s-a+1)}$dx
$B(e^{ax}g(x))$ = B (s-a)
Hence the proof

3.4 Theorem
  If g(x) is a piecewise continuous function on[0,∞], but not of exponential order, then a polynomial integral transform.
$B(g(x)) \to 0$, As s $\to \infty$

Proof
Let
$B(g(x)) = \int_1^\infty 1$ g(ln x) x^{-s-1}dx
$|B(g(x))| \le \int_1^\infty 1$ |g(ln x) x^{-s-1}dx|
$B(g(x)) = \int_1^\infty 1$ g(ln x) $|x^{-s-1}|$dx
But we observe that $x^{-(s+1)} \to 0$, s $\to \infty$.
It follows that B(g(x)) $\to 0$, s $\to \infty$.
                          Hence the proof.

4.The Polynomial Integral Transform of Derivative
In this section , we give the polynomial integral transform of derivatives of the function g(x) with respect to x. 4.1

4.1Theorem
If   g, g^1,…, g^{n-1} are continuous on [1,∞], and if f^n(x) is piecewise continuous on[1,∞] , then
$B(g^{(n)}(x))$ = s^nG(s) - s^{n-1} g(0) - s^{n-2} g(0) -…-g^{n-1}(0)
Where G(s) = B(g(x))

Proof
Let B$(g'(x))$ = = $\int_1^\infty 1$ g'(ln x) $|x^{-s-1}|$dx
Using integration by parts, we obtain

Let L$(g'(x))$ = $\lim[_{t \to 0} g \ln x)$ $x^{-s}]_1^\infty$ + $\int_1^t g(lnx)\frac{1}{x}x^{-s}$dx
    L$(g'(x))$ = sG(s)- g(0)
Proceeding a similar as above , we obtain
$B(g''(x)) = \int_1^\infty g'(lnx)$ x^{-s-1} dx $\to$ B$(g''(x))$ = -$g'(0)$ + sL$(g'(x))$
Substituting the expression of g'(x) into the above equation , we obtain

$B(g''(x)) = s^2 G(x) - sg(0) - g'(0)$

By induction, we obtain
$B(g^{(n)}(x)) = s^n G(s) - s^{n-1} g(0) - s^{n-2} g'(0) - \ldots - g^{n-1}(0)$

Where $G(s) = B(g(x))$.

Hence the proof.

Conclusion

We observed that the polynomial integral transform solves differential equation with a few computations as well as time. Unlike the Laplace integral transform, the polynomial integral transform involves a polynomial function as its kernel, which is easier and transforms complicated functions into algebraic equations. The solution of the differential equation is then obtained from the algebraic equation.

 Also, using the polynomial integral transform, the convergence of the solution of the differential equation is faster as compared with the Laplace integral transform and others. We observed that the Polynomial Integral Transform is defined on the interval $[1, \infty]$.

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