UNCONSTRAINED OPTIMIZATION USING INTERVAL ARITHMETIC AND INTERVAL ORDER RELATIONS

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ABSTRACT

This paper deals with discussion on the interval analysis, finite interval arithmetic, interval order relations and its application to find the global optimal solution of unconstrained optimization problems. The existing works on the interval order relations have been pointed out and the modified definitions have been presented. An interval technique is proposed to solve unconstrained unimodal/multimodal optimization problems with continuous variables. In this proposed method, the search region is divided into two equal sub-regions successively and in each sub-region, the lower and upper bounds of the objective function are computed with the help of interval arithmetic. Afterward, interval order relations are used to compare these two interval valued objectives and considering the sub-region containing the better objective value, the global optimal value or close to it of the objective function is obtained. As a final point, the proposed technique is applied to solve a number of test problems of global optimization with lower as well as higher dimensions available in the literature and is compared with the existing methods with respect to the number of function evaluations.

Keyword: - *Global Optimization, Interval Number, Sub-region, Interval Order Relations, Optimistic and Pessimistic Point of View, Unconstrained Optimization*

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1. INTRODUCTION

In the existing literature of global optimization, the most of the researcher have done their works and have proposed different techniques which either use the derivative information or not. Some researchers have used several soft computing techniques with the combination of several evolutionary search techniques, like, genetic algorithm (GA), particle swarm optimization (PSO), ant colony algorithm (ACO), differential evolution (DE), etc. Several researchers have also been used simulation based search techniques to find the global optimal solution of constrained as well as unconstrained optimization problems. Sometimes they have optimized the problems in precise environment and sometimes in imprecise environment.

While considering the optimization problems as decision making problems, specially, in case of heuristic search methods, the selection of the better or the best value of the objective function is of formidable task. It is also very difficult, in the case when objective function is interval valued, as is considered in this work. Here, a set of intervals appear in the selection of the best choice. This raises a question regarding the comparison of two arbitrary interval numbers. To find the better interval, Moore [12] defined two transitive order relations of interval numbers. However, these order relations cannot find the ranking between two partially or fully overlapping interval numbers. After Moore [12], Ishibuchi and Tanaka [9] recommended two order relations "$\leq_{LR}$" and "$\leq_{CW}$." After them, Levin [10] defined a remoteness function to compare two arbitrary interval numbers. However, this process for comparison is very much complicated to find out the best choice. Then, Sevastjanov and Róg [16] anticipated in the probabilistic approach.
Decision-making is a significant job for the choosing the best choice in conflicting situations. It depends upon the uncertainty of the problem coming out from different behavior of the parameters involved and also from the decision

variables. There are two types of decision making, namely, the optimistic and the pessimistic decision-making. For optimistic decision-making, the decision-maker chooses the best option ignoring the uncertainty whereas, in the pessimistic case, the decision-maker selects the best option with less uncertainty.

In this context, the solution procedure is extremely significant factor for finding global optima of a multimodal, multidimensional, non-convex, non-linear, continuous optimization problem with fixed coefficients. There are several deterministic and stochastic methods proposed for finding the global optima or a value near to it. These methods are available in Floudas et al. [6] and Hansen and Walster [7]. Ichida [8] developed an interval computing method to find out the global optima of the problems with fixed coefficients. In this interval computing method, the search domain is divided into two sub-regions and the lower and upper bounds of the objective function are estimated in each sub-region. By rejection principle, we can reject one of the sub-regions. Continuing this process, we can find the global optima. There are several rejection principles for rejection of one sub-region from two. In this connection, one may refer to the works of Csendes [5], Markót et al. [11], Csallner et al. [4], and Casado et al. [2]. In this paper, we have studied the existing works on comparing and ranking of any two interval numbers. After pointing out the weakness of these definitions, a new approach is suggested in the context of decision maker's (optimistic and pessimistic) point of view. After that, the definition of interval order relations irrespective of decision makers' point of view is considered [23].

Secondly, an interval computing technique is proposed to solve nonlinear bound constrained (also known as the box constrained) optimization problems. In the interval computing technique, the original domain of variables is divided into two equal sub-regions successively and the lower and upper bounds of the objective function are computed in each sub-region with the help of interval arithmetic. Now, by comparing these two interval objective values by the proposed order relations in [23], and then considering the sub-region containing the better objective value, one can always find out the global optimal value of the objective function or close to it in the form of an interval with negligible width. In conclusion, this method is tested on several test functions available in the literature and is compared with the existing methods with respect to the number of function evaluations.

2. NOTATIONS USED

The notations which have been used in this paper are given in the table below

Table -1: Notations Used

Notation	Description
$A=[a_L, a_R]$	Interval number A
a_L	Left bound of interval number A
a_R	Right bound of interval number A
a_C	Centre of interval number A
a_W	Radius/half width of interval number A
$Z=f(x)$	Objective function
$x=(x_1,x_2,...,x_n)$	Decision vector
l	Lower bound of decision vector
u	Upper bound of decision vector
l_j, u_j	Lower and upper bounds of variable x_j
D	Search domain
R^n	*n* dimensional space
$F(R_k)=[\underline{f_k}, \overline{f_k}]$	Interval valued objective function in the sub-region R_k
$\underline{f_k}, \overline{f_k}$	Lower and upper bounds of f(x) in the selected search sub-region R_k in the *k*-th iteration

3. ASSUMPTIONS

The assumptions which have been taken up are,

i) Interval numbers assumed to finite and closed intervals
ii) The test functions are defined and continuous in the given domain
iii) All the test functions have their extremas in the search domain.
iv) The search domain is bounded.
v) The test functions may be convex/non-convex, continuous, unimodal/multimodal.
vi) The test functions may have one or more variables.

4. FINITE INTERVAL ARITHMETIC

An interval number $A=[a_L,a_R]$ is defined to be the closed interval $A=[a_L,a_R]=\{x: a_L \le x \le a_R, x \in \mathbb{R}\}$, where a_L, a_R are the lower and upper bounds respectively and $\mathbb{R}$ is the set of all real numbers. The interval number $A=[a_L,a_R]$ can also be represented in the centre and the radius form as $A=\langle a_c, a_w\rangle$, where $a_c=(a_L+a_R)/2$ and $a_w=(a_R-a_L)/2$ be the centre and radius of the interval *A*. It is to be noted that every real number $x \in \mathbb{R}$ can also be treated as a degenerate interval $[x,x]$ of zero width. The works of Hansen and Walster [7] and Mahato and Bhunia [22] and Karmakar et al. [21] may be referred for details regarding interval arithmetic, integral power of interval number and also the *n*-th root as well as the rational power of interval number.

Definitions: Let $A=[a_L,a_R]$ and $B=[b_L,b_R]$ be two intervals. Then the definitions of addition, subtraction, scalar multiplication, multiplication and division of interval numbers are as follows:

Addition of two interval numbers A and B: $A+B=[a_L,a_R]+[b_L,b_R]=[a_L+b_L, a_R+b_R]$.

Subtraction of an interval number B from another one A:
$A-B=[a_L,a_R]-[b_L,b_R]=[a_L,a_R]+[-b_R,-b_L]=[a_L-b_R, a_R-b_L]$.

Multiplication of an interval number A by any real number k: For any real number *k*,

$$kA=k[a_L,a_R]=\begin{cases}[ka_L,ka_R] \text{ if } k\ge 0\\ [ka_R,ka_L] \text{ if } k<0.\end{cases}$$

Multiplication of two interval numbers A and B:
$A\times B=[a_L,a_R]\times[b_L,b_R]=[\min(a_Lb_L, a_Lb_R, a_Rb_L, a_Rb_R), \max(a_Lb_L, a_Lb_R, a_Rb_L, a_Rb_R)]$.

Division of an interval number A by another one B: $\frac{A}{B}=A\times\frac{1}{B}=[a_L,a_R]\times[\frac{1}{b_R},\frac{1}{b_L}]$, provided $0\notin[b_L,b_R]$.

Positive integral power of an interval number A: Let $A=[a_L,a_R]$ be an interval then for any non-negative integer *n*,

$$A^n=\begin{cases}[1,\ 1] & \text{if } n=0\\ [a_L^n,\ a_R^n] & \text{if } a_L\ge 0 \text{ or if } n \text{ is odd}\\ [a_R^n,\ a_L^n] & \text{if } a_R\le 0 \text{ and } n \text{ is even}\\ [0,\ \max(a_L^n,\ a_R^n)] & \text{if } a_L\le 0\le a_R \text{ and } n(>0) \text{ is even.}\end{cases}$$

The *n*-th root of an interval number A: The *n*-th root of an interval $A=[a_L,a_R]$, n being a positive integer is defined as,

$$A^{\frac{1}{n}} = \begin{cases} [\sqrt[n]{a_L},\ \sqrt[n]{a_R}] & \text{if } a_L \ge 0 \text{ or if } n \text{ is odd} \\ [0,\ \sqrt[n]{a_R}] & \text{if } a_L \le 0 \le a_R \text{ and } n(>0) \text{ is even} \\ \phi & \text{if } a_R < 0 \text{ and } n \text{ is even.} \end{cases}$$

where ϕ is the empty interval.

The modulus of an interval number A: The modulus of an interval $A=[a_L, a_R]$ is defined as,

$$|A| = \left|[a_L, a_R]\right| = \begin{cases} [a_L, a_R] & \text{if } a_L \ge 0 \\ [|a_R|, |a_L|] & \text{if } a_R \le 0 \\ [0, |a_L|] & \text{if } a_L < 0,\ a_R > 0, |a_L| \ge |a_R| \\ [0, |a_R|] & \text{if } a_L < 0,\ a_R > 0, |a_L| < |a_R|. \end{cases}$$

4.1 FUNCTIONS OF INTERVALS

Some functions of interval which have been included in the test functions considered are presented here. These important functions are exponential function, logarithmic function and bounded trigonometric functions.

EXPONENTIAL FUNCTION

As the exponential function $f(x) = e^{ax}$ is monotonic over any interval, the exponential extension of the interval $A=[a_L, a_R]$ is defined as

$$\exp(A) = \exp([a_L, a_R]) = \left[\exp(a_L), \exp(a_R)\right],\quad \exp(-A) = \exp(-[a_L, a_R]) = \exp([-a_R, -a_L]) = \left[\exp(-a_R), \exp(-a_L)\right]$$

LOGARITHMIC FUNCTION

As the exponential function $\log(x)$, for $x > 0$ is monotonic over any interval, the logarithmic extension of the interval $A=[a_L, a_R]$ is defined as

$$\log(A) = \log([a_L, a_R]) = \left[\log(a_L), \log(a_R)\right], \text{ provided } a_L > 0.$$

SINE AND COSINE FUNCTIONS

The trigonometric functions *sin(A)* and *cos(A)* can be evaluated over any given interval A by evaluating the values of the functions at the end points and checking whether the interval contains a point or points where *sin(A)* and *cos(A)* can have extreme values.

$$\sin([a_L, a_R]) = \left[b_L, b_R\right]$$

where

$$b_L = \begin{cases} -1 & \text{if } \exists\, k \in \square :\ 2k\pi - \dfrac{\pi}{2} \in \left[a_L, a_R\right] \\ \min\left\{\sin(a_L), \sin(a_R)\right\} & \text{otherwise} \end{cases}$$

$$\text{and } b_L = \begin{cases} 1 & \text{if } \exists\, k \in \square :\ 2k\pi + \dfrac{\pi}{2} \in \left[a_L, a_R\right] \\ \min\left\{\sin(a_L), \sin(a_R)\right\} & \text{otherwise.} \end{cases}$$

The function $\cos([a_L, a_R])$ can be defined similarly.

4.2 ORDER RELATIONS OF INTERVAL NUMBERS

In this section, the development of order relations of interval numbers has been discussed. Any two arbitrary interval numbers $A=[a_L,a_R]$ and $B=[b_L,b_R]$ may be categorized into the following three types (Fig.-1):

Type I: Non-overlapping i.e., the intervals are completely disjoint.
Type II: Partially overlapping intervals.
Type III: Fully overlapping intervals i.e., one of the intervals contains the other.

In this context, Moore [12] first pointed out two transitive order relations of the interval numbers. For any two intervals $A=[a_L,a_R]$ and $B=[b_L,b_R]$, he gave the first transitive order relation '<' as

$$A<B \text{ iff } a_R<b_L$$

and the other transitive order relation for intervals is the set inclusion property. This is depicted as A⊆B iff $b_L<a_L$ and $a_R<b_R$.

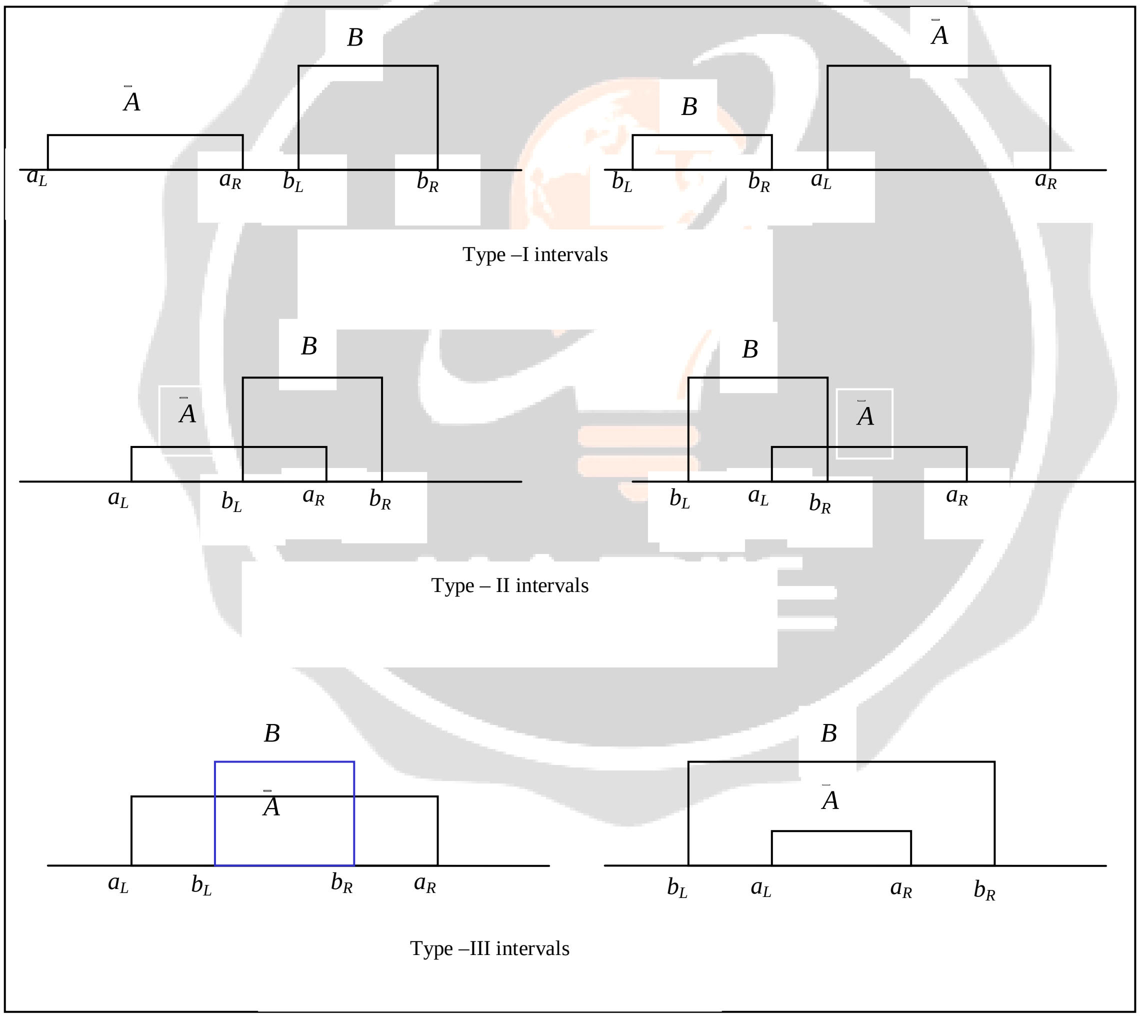

Figure-1: Different types of interval numbers

Then, Ishibuchi and Tanaka [9] defined the order relations of two closed intervals $A=[a_L,a_R]=\langle a_c,a_w\rangle$ and $B=[b_L,b_R]=\langle b_c,b_w\rangle$, in the following ways:

(i) $A\leq_{LR} B$ iff $a_L \leq b_L$ and $a_R \leq b_R$
$A<_{LR} B$ iff $A\leq_{LR} B$ and $A\neq B$

(ii) $A\leq_{cw} B$ iff $a_c \leq b_c$ and $a_w \leq b_w$
$A<_{cw} B$ iff $A\leq_{cw} B$ and $A\neq B$.

These order relations are reflexive, transitive and anti-symmetric i.e., partially order relations. Clearly, for a minimization problem, the decision maker will prefer the interval *A*. Generalizing the definitions of Ishibuchi and Tanaka [9], in1996, Chanas and Kuchta [3] proposed the concept of t_0t_1 – cut of an interval and defined new order relations.

Kundu [24] first noticed that the interval ranking methods discussed above could not find the measure 'How much larger the interval is, if the interval is greater than the other?' Introducing the 'fuzzy leftness relation' he attempted to answer this question. For the intervals A and B, let $x \in A$ and $y \in B$ be uniformly and independently distributed in A and B respectively. Then *A* is left to *B* if Left $(A,B) = max\{0,P(x < y) - P(x > y)\} > 0$ and *A* is right to B if Right $(A,B) = max\{0,P(x > y) - P(x < y)\} > 0$ where $P(x < y)$ denotes the probability that $x < y$. This is a probabilistic approach.

In the year, 2000, another two approaches of ranking of two closed intervals were given in [15]. In the first approach, they defined the acceptability function (or acceptability index or value judgment index) $\Lambda: I\times I \rightarrow [0,\infty)$ for the intervals A and B as

$$\Lambda(A,B)=\frac{b_c-a_c}{b_w+a_w},\ \text{where } b_w+a_w\neq 0.$$

This may be regarded as a grade of acceptability of the 'first interval to be inferior to the second'. If $\Lambda(A,B)=0$, then for the minimization problem, the interval A cannot be accepted. If $0<\Lambda(A,B)<1$, A can be accepted with the grade of acceptability $(b_c-a_c)/(b_w+a_w)$. Again, for $\Lambda(A,B)=1$, A is accepted with full satisfaction. According to them, the acceptability index is only a value based ranking index and it can be applied partially to select the best alternative from the pessimistic point of view of the decision maker. So, only the optimistic decision maker can use it completely. In another approach, Sengupta and Pal [15] introduced the fuzzy preference ordering for the ranking of a pair of interval numbers on the real line with respect to a pessimistic decision maker's point of view.

Mahato and Bhunia [22] presented the modified definitions of ranking with respect to optimistic and pessimistic decision makers' point of view for maximization and minimization problems separately. In 2012, Sahoo et al. [23] proposed the simplified definition of interval order relations ignoring optimistic and pessimistic decisions. It is to be mentioned that both the definitions by Mahato and Bhunia [22] and Sahoo et al. [23] report the same result. The latest most generalized interval order relations irrespective of optimistic and pessimistic decision makers' point of view are given below for maximization and the minimization problems separately.

Interval ranking for maximization problem: Let $A=[a_L,a_R]=\langle a_c,a_w\rangle$ and $B=[b_L,b_R]=\langle b_c,b_w\rangle$ be two intervals. Then if

$$A>_{\max} B \Leftrightarrow \begin{cases} a_c>b_c \text{ for Type 1 and Type 2 intervals} \\ \text{either } a_c\geq b_c \wedge a_w<b_w \text{ or } a_c\geq b_c \wedge a_R>b_R \text{ for Type 3 intervals,} \end{cases}$$

the interval A is accepted for maximization problems. The order relation "$>_{\max}$" is reflexive, transitive but not symmetric.

Interval ranking for minimization problem: Let $A=[a_L,a_R]=\langle a_c,a_w\rangle$ and $B=[b_L,b_R]=\langle b_c,b_w\rangle$ be two intervals. Then if

$$A <_{\min} B \Leftrightarrow \begin{cases} a_c < b_c \quad \text{for Type 1 and Type 2 intervals} \\ \text{either } a_c \le b_c \wedge a_w \le b_w \ \text{ or } a_c \le b_c \wedge a_L < b_L \ \text{for Type 3 intervals,} \end{cases}$$

the interval A is accepted for minimization problems. The order relation "$<_{\min}$" is reflexive, transitive but not symmetric.

5. SOLUTION PROCEDURE

Let us consider an unconstrained optimization (maximization or minimization) problem with fixed coefficients as follows:

$Z = f(x),\ l \le x \le u,$

where, $x = (x_1, x_2, ..., x_n), l = (l_1, l_2, ..., l_n), u = (u_1, u_2, ..., u_n)$ and n represents the number of variables, x_j is the j-th decision variable whose prescribed upper and lower bounds are l_j and u_j, respectively. Hence, the search region of the above problem is as follows:

D ={x ∈ R^n : $l_j \le x_j \le u_j$, $j= 1,2,...,n$}.

5.1 INTERVAL METHODOLOGY

The prescribed domain is defined as

$D = \{x \in R^n : l_j \le x_j \le u_j,\ j = 1,2,...,n\}$.

Then, D can be divided into two sub-regions R_1 and R_2 with respect to the variable x_k(k = $1,2,...,n$) defined as follows:

$$R_1 = \left\{ x \in R^n : l_k \le x_k \le \frac{l_k + u_k}{2}, l_j \le x_j \le u_j; j = 1,2,...,k-1,k+1,...,n \right\},$$

$$R_2 = \left\{ x \in R^n : \frac{l_k + u_k}{2} \le x_k \le u_k, l_j \le x_j \le u_j; j = 1,2,...,k-1,k+1,...,n \right\}.$$

Let $F(R_1) = \left[\underline{f_1}, \overline{f_1} \right]$ and $F(R_2) = \left[\underline{f_2}, \overline{f_2} \right]$ be the interval values of $f(x)$ in the sub-regions R_1 and R_2, respectively, where $\underline{f_k}, \overline{f_k}$ $(k = 1,2)$ denote the lower and upper bounds of $f(x)$ in R_k, calculated by applying interval arithmetic. Then comparing $F(R_1)$ and $F(R_2)$, the sub-region either R_1 or R_2 that contains the better objective value, is accepted. This process for each sub-region is repeated till the domain of each variable reduces to an interval with negligible width. Finally, the global optimal value or close to the optimal value of the given objective function has been obtained. For the entire process, Algorithm 6.1 is developed for minimization/maximization problems.

Algorithm 5.1

Step-1: Initialize the number of variables n.

Step -2: Initialize the lower and upper bounds l_j and u_j ($j = 1,2,...,n$)of the variables.

Step-3: Compute the widths $\varepsilon_j = u_j - l_j$; $j = 1,2,...,n$.

Step- 4: If $\varepsilon_j < a$,a pre-assigned very small positive number, then go to Step(7); otherwise go to the next step.

Step-5: (i) Divide the accepted sub-region or region X into two other smaller distinct sub-regions R_1 and R_2 such that $R_1 \cup R_2 = X$.

(ii) Applying interval arithmetic, compute the interval value $F(R_1) = \left[\underline{f_1}, \overline{f_1} \right]$ and $F(R_2) = \left[\underline{f_2}, \overline{f_2} \right]$ *of the objective function in the sub-regions R_1 and R_2, respectively.*

(iii) Select the sub-region R_1 or R_2 as the new search region which contains the better objective function value by comparing $F(R_1)$and $F(R_2)$with the help of order relations between two intervals defined for minimization and maximization problems, respectively.

Step -6: Go to Step 3.

Step- 7: Print the values of the variables and the objective function in the form of intervals.

Step- 8: Stop.

6 NUMERICAL EXAMPLES

Numerical experiments have been carried out to test the performance of the proposed approach described in this paper. A number of well-known test functions have been selected from the literature [2,13,14]. These test functions have different features like convex/non-convex, continuous, unimodal/multimodal and low/high dimension. Solving these test problems, the global optimal solution or near to optimal have been found by applying Algorithm 6.1. The test problems with their results have been given in the appendix with corresponding number of function evaluation with error tolerance $\varepsilon = 10^{-8}$. Tables 2-7 also show the comparison in number of function evaluations among the available methods, like, TIAM (Traditional Interval analysis global minimization Algorithm with Monotonicity test), IAG (Interval analysis global minimization Algorithm using Gradient information), GA-SQ (Genetic Algorithm by Salhi and Queen [14]), and the proposed method in this paper. The approach for computing the best-found value in each sub-region of a given search region of the test problem has been coded in C programming language and implemented on a Core 2 Duo PC, 4.0 GHz with 2 GB RAM in LINUX environment.

Table-2: Computational results & comparison of the proposed method with others methods for **single variable** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$f(x)=2x^2-0.03\exp\left\{-(200(x-0.0675))^2\right\}$, $-10\le x\le 10$	[0.067388, 0.067388]	[-0.020903,-0.020903]	112	88	-	62
2	$f(x)=\sin(x)+\sin(\frac{10x}{3})+\ln(x)-0.84x$, $2.7\le x\le 7.5$	[5.199778, 5.199778]	[-4.601308,-4.601308]	132	81	-	58
3	$f(x)=\sin^2(1+\frac{x-1}{4})+(\frac{x-1}{4})^2$, $-10\le x\le 10$	[-0.787880, -0.787880]	[0.475689,0.475689]	167	114	-	62
4	$f(x)=(x-1)^2(\sin^2(1+x))+1$, $-10\le x\le 10$	[1.000000, 1.000000]	[1.000000, 1.000000]	78	62	-	62
5	$f(x)=-\frac{1}{(x-2)^2+3}$, $-10\le x\le 10$	[2.000000, 2.000000]	[-0.333333, -0.333333]	66	56	-	62
6	$f(x)=(x-x^2)^2(x-1)^2$, $-10\le x\le 10$	[1.000000, 1.000000]	[0.000000, 0.000000]	107	88	-	62
7	$f(x)=\exp(x^2)$, $-10\le x\le 10$	[0.000000, 0.000000]	[1.000000, 1.000000]	199	116	-	62
8	$f(x)=-\sum_{k=1}^{5}k\sin((k+1)x+k)$, $-10\le x\le 10$	[-6.774576, -6.774576]; [-0.491391, -0.491391]; [5.791794, 5.791794]	[-12.031249, -12.031249]	459	333	-	62
9	$f(x)=\frac{x^2}{20}$-cos(x)+2, $-20\le x\le 20$	[0.000000, 0.000000]	[1.000000, 1.000000]	207	114	-	64
10	$f(x)=-\sum_{i=1}^{10}\frac{1}{(k_i(x-a_i))^2+c_i}$, $0\le x\le 10$ a=(3.040,1.098,0.674,3.537,6.173,8.679,4.503,3.328,6.937,0.700) k=(2.983,2.378,2.439,1.168,2.406,1.236,2.868,1.378,2.348,2.268) c=(0.192,0.140,0.127,0.132,0.125,0.189,0.187,0.171,0.188,0.176)	[0.685844, 0.685844]	[-14.572917, -14.572917]	139	113	-	60

Table-3: Computational results & comparison of the proposed method with others methods for **two variables** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$R2(x_1,x_2)=100(x_2^2-x_1)^2+(1-x_1)^2$, $-2\le x_1,x_2\le 2$	x_1=[1.000000, 1.000000] x_2=[1.000000, 1.000000]	[0.000000, 0.000000]	-	-	200	116
2	$ES(x_1,x_2)=-\cos(x_1)\cos(x_2)e^{-[(x_1-\pi)^2+(x_2-\pi)^2]}$, $-100\le x_1,x_2\le 100$	x_1=[3.141593, 3.141593] x_2=[3.141593, 3.141593]	[-1.000000, -1.000000]	-	-	-	140
3	$BH(x_1,x_2)=x_1^2+2x_2^2-0.3\cos(3\pi x_1)\cos(4\pi x_2)+0.3$, $-100\le x_1,x_2\le 100$	x_1=[0.000000, 0.000000] x_2=[0.000000, 0.000000]	[0.000000, 0.000000]	-	-	-	140

Table-4: Computational results & comparison of the proposed method with others methods for **three variables** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$F1(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2$, $-100\le x_1,x_2,x_3\le 100$	x_i=[0.000000, 0.000000] *i=1,2,3*	[0.000000, 0.000000]	-	-	-	180
2	$R_3(x)=\sum_{j=1}^{2}[100\left(x_j^2-x_{j+1}\right)^2+\left(x_j-1\right)^2]$ $-2\le x_i\le 2,\ \ i=1,2,3$	x_i=[1.000000, 1.000000] *i=1,2,3*	[0.000000, 0.000000]	-	-	-	180

Table-5: Computational results & comparison of the proposed method with others methods for **five variables** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$F1(x)=\sum_{i=1}^{5}x_i^2$, $-100\le x_i\le 100,\ i=1,2,...,5$	x_i=[0.000000, 0.000000] *i=1,2,…,5*	[0.000000, 0.000000]	-	-	-	300
2	$R_5(x)=\sum_{j=1}^{4}[100\left(x_j^2-x_{j+1}\right)^2+\left(x_j-1\right)^2]$ $-2\le x_i\le 2,\ \ i=1,2,...,5$	x_i=[1.000000, 1.000000] *i=1,2,…,5*	[0.000000, 0.000000]	-	-	-	280

Table-6: Computational results & comparison of the proposed method with others methods for **ten variables** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$F1(x)=\sum_{i=1}^{10}x_i^2$, $-100\le x_i\le 100,\ i=1,2,...,10$	x_i=[0.000000, 0.000000] *i=1,2,…,10*	[0.000000, 0.000000]	-	-	-	600
2	$R_{10}(x)=\sum_{j=1}^{9}[100\left(x_j^2-x_{j+1}\right)^2+\left(x_j-1\right)^2]$ $-2\le x_i\le 2,\ \ i=1,2,...,10$	x_i=[1.000000, 1.000000] *i=1,2,…,10*	[0.000000, 0.000000]	-	-	-	560

Table-7: Computational results & comparison of the proposed method with others methods for **fifty variables** test functions

Sl. No.	Test Problem	Optimal Solution	Min. Objective Value	No. of Function Evaluations			
				TIAM	IAG	GA-SQ	Proposed Method
1	$F1(x)=\sum_{i=1}^{50} x_i^2$, $-100 \le x_i \le 100,\ i=1,2,\dots,50$	x_i=[0.000000, 0.000000] *i=1,2,…,50*	[0.000000, 0.000000]	-	-	-	3000
2	$R_5(x)=\sum_{j=1}^{49}[100\left(x_j^2-x_{j+1}\right)^2+\left(x_j-1\right)^2]$ $-2 \le x_i \le 2,\ i=1,2,\dots,50$	x_i=[1.000000, 1.000000] *i=1,2,…,50*	[0.000000, 0.000000]	-	-	-	2800

7 CONCLUDING REMARKS

We have reported a technique to find the global solution of unconstrained optimization problems of different types. The proposed method is based on interval computing technique and interval order relations. In this technique, the values of the interval valued objective function are computed in each of the sub-regions and then the better one is chosen. Then, the sub-region having better objective value is again subdivided into two sub-regions and the process continued. The definitions of bounded trigonometric function of an interval are given. To illustrate the performance of the technique, some numerical examples have been solved and the results are also presented. The test functions are available in the literature and they are of single variable as well as several variables. From the numerical results, it has been seen that the proposed method possesses the merits of global exploration, fast convergence, and it can find the optimal or close-to-optimal solutions. This technique, which has been applied for finding the best value in each sub-region, gives the better result for smaller sub-region. For future research, one may apply the same methodology for other type of optimization in operations research.

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