Ulam Stability of Linear Differential Equationwith Initial and Boundary Conditions

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ABSTRACT- We purpose of this article, we investigate to prove the Hyers – Ulam and Hyers – Ulam-Rassias stability of then^{th}order Linear Differential equation with initial and boundary conditions Applying Taylor's formula.

Index Terms- Hyers – Ulam and Hyers – Ulam-Rassias stability, initial and boundary conditions, Taylor's formula.

1.INTRODUCTION

The study of stability for a variety of functional equations is invented by a famous mathematician S. M. Ulam [23] in 1940. He raised the question concerning the stability of functional equations: "Give Conditions in order for a linear function near an approximately linear function to exist". The first solution was brilliantly answered to the problem of Ulam for Cauchy additive functional equation based on Banach spaces by D. H. Hyers [24] in 1941. A generalized result to Ulam's question for approximately linear mappings was proposed by Th. M. Rassias [25] in 1978. During these days, a lot of mathematicians contributed to the expansion of the Ulam's problem with other functional equations with other spaces in different directions. [3-5, 8-11, 14-17, 28].

UHsisnow day proposed for replacing functional equations with differential equations: The differential equation

$$\zeta\left(g,\varphi,\varphi',\varphi'',...,\varphi^{(n)}\right)=0$$

is said to have HUs if for a given $\grave{o}>0$ and a function φ such that

$$\left|\zeta\left(g,\varphi,\varphi',\varphi'',...,\varphi^{(n)}\right)\right|\le\grave{o},$$

there exists some φ_a of the differential equation such that $|\varphi(s)-\varphi_a(s)|\le H(\grave{o})$ and $\lim_{\grave{o}\to 0}H(\grave{o})=0.$

Oblaza seems to be the first author who has investigated the HUs of linear differential equations [12, 13]. Thereafter [18], which exist the HUs of the linear differential equation $\psi'(l)=\lambda\psi(l)$. S. M. Jung continuously posed the general setting for HUs of first order linear differential equations in [30-32].

The matrix method and HUs of a system of linear differential equations with a co-efficient posed by S. M. Jung [33] in 2006. The Generalized HUs of higher order linear differential equation Applying the Laplace transform method proved by Alqifiary and Jung [12] in 2014.

In recent years, J.M. Rassias *et al.*[22] proved the Mittag-Leffler-HUs of first order and second order linear differential equations Applying Fourier transform method. Then recently, J.M. Rassias [1] investigated the Mittag-Leffler-HUs of first order and second order nonlinear initial value problems Applying the Laplace transform method. HUs of differential equations is now being studied [2, 6, 7, 19-21, 26, 27, 29] and the investigation is ongoing.

In this article, we study the UHs and UHRs of then^{th}order Linear Differential equation of the form

$$x^{(n)}(s)+a_n(s)x(s)=0 \tag{1}$$

with boundary conditions

$$x(a)=x(b)=0 \tag{2}$$

and with initial conditions

$$x(a)=x'(a)=x''(a)=\ldots=x^{(n-1)}(a)=0 \tag{3}$$

where $x\in C^n[a,b], a_n(s)\in C[a,b]$, $-\infty<a<b<\infty$.

I. Preliminaries

Here, we recall the definition of the HUs property with boundary conditions and initial conditions.

Definition 2.1. The Eq.(1.1) has the HUs property with boundary conditions (1.2), if there exists a constant $K > 0$ with the following property, for every $\epsilon > 0$, $x \in C^n\ [a,b]$, if $\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon$ and $x(a) = x(b) = 0$, then there exists some $y \in C^n\ [a,b]$, satisfying $y^{(n)}(s) + a_n(s)y(s) = 0$ and $y(a) = y(b) = 0$, such that $|x(s) - y(s)| \leq K\epsilon$. We call such K as a HUs constant for the equation (1.1).

Definition 2.2. The Eq.(1.2) has the HUs property with initial conditions (1.3), if there exists a constant $K > 0$ with the following property, for every $\epsilon > 0$, $x \in C^n\ [a,b]$, if $\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon$ and $x(a) = x'(a) = x''(a) = \ldots = x(n-1)(a) = 0$, then there exists some $y \in C^n\ [a,b]$, satisfying $y^{(n)}(s) + a_n(s)y(s) = 0$ and $y(a) = y(b) = 0$, such that $|x(s) - y(s)| \leq K\epsilon$. We call such K as a HUs constant for the equation (1.1).

Definition 2.3. The Eq.(1) has the HURs with $\theta(t)$, where, $\theta : R \to [0,\infty)$ and with the boundary condition (2), if there exists a positive constant $K > 0$ such that the following property holds: For every $\epsilon > 0$, $x \in C^n\ [a,b]$, if $\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon\,\phi(s)$ and $x(a) = x(b) = 0$, then there exists some $y \in C^n\ [a,b]$, satisfying $y^{(n)}(s) + a_n(s)y(s) = 0$ and $y(a) = y(b) = 0$, such that $|x(s) - y(s)| \leq K\,\epsilon\,\phi(s)$. We call such K as a HURs constant for the equation (1.1).

Definition 2.4. The Eq.(1) has the HURs with $\theta(t)$, where, $\theta : R \to [0,\infty)$ and with the initial condition (3), if there exists a positive constant $K > 0$ such that the following property holds: For every $\epsilon > 0$, $x \in C^n\ [a,b]$, if $\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon\,\phi(s)$ and $x(a) = x'(a) = x''(a) = \ldots = x(n-1)(a) = 0$, then there exists some $y \in C^n\ [a,b]$, satisfying $y^{(n)}(s) + a_n(s)y(s) = 0$ and $y(a) = y(b) = 0$, such that $|x(s) - y(s)| \leq K\,\epsilon\,\phi(s)$. We call such K as a HURs constant for the equation (1.1).

II. HUs and HURs of the differential equation.

Now, in the following Theorems, we prove the HUs and HURs property with boundary conditions and with initial conditions of the differential equation.

Theorem 3.1.

If $max\ |a^n(s)| < \frac{n!\,2^n}{(b-a)^n}$. Then the equation (1.1) has the HUs stability property with boundary conditions (1.2).

Proof.

For every $\epsilon > 0, C^n\ [a,b]$, if $x^{(n)}(s) + a_n(s)x(s) \leq \epsilon$ and $x(a) = x(b) = 0$. Let $M = max\{|x(s)| : s \in [a,b]\}$, since $x(a) = x(b) = 0$, there exists $t_0 \in (a,b)$ such that $|x(s_0)| = M$. Then by Taylor's formula, we obtain

$$x(a) = x(s_0) + x'(s_0)(s_0 - a) + \frac{x''(s_0)}{2!}(s_0 - a)^2 + \ldots + \frac{x^{(n)}(\alpha)}{n!}(s_0 - a)^n \tag{3.1}$$

$$x(b) = x(s_0) + x'(s_0)(b - s_0) + \frac{x''(s_0)}{2!}(b - s_0)^2 + \ldots + \frac{x^{(n)}(\beta)}{n!}(b - s_0)^n \tag{3.2}$$

where $\alpha \in (a, s_0)$ and $\beta \in (s_0, b)$. Since we obtain $x(a) = 0$, then (3.1) becomes

$$\frac{x^{(n)}(\alpha)}{n!}(s_0 - a)^n + \cdots + \frac{x''(s_0)}{2!}(s_0 - a)^2 + x(s_0) + x'(s_0)(s_0 - a) + x(s_0) = 0$$

thus we obtain

$$\left\|x^{(n)}(\alpha)\right\| = \frac{M\,n!}{(s_0 - a)^n}\ .$$

Since we obtain $x(b) = 0$, then (3.2) becomes

$$\frac{x^{(n)}(\beta)}{n!}(b - s_0)^n + \cdots + \frac{x''(s_0)}{2!}(b - s_0)^2 + x'(s_0)(b - s_0) + x(s_0) = 0$$

thus we obtain

$$\left\|x^{(n)}(\beta)\right\| = \frac{M\,n!}{(s_0 - b)^n}$$

On the other hand, let $s_0 \in (a, \frac{a+b}{2}]$, we obtain

$$\frac{M\,n!}{(s_0 - a)^n} \geq \frac{M\,n!}{\frac{(b-a)^n}{2^n}} = \frac{2^n\ M\,n!}{(b - a)^n}$$

$$\frac{M\,n!}{(s_0 - a)^n} \geq \frac{2^n\ M\,n!}{(b - a)^n} \tag{3.3}$$

and on the other case $s_0 \in (\frac{a+b}{2}, b\ ]$,we obtain

$$\frac{M\,n!}{(s_0 - b)^n} \geq \frac{M\,n!}{\frac{(b-a)^n}{2^n}} = \frac{2^n\ M\,n!}{(b-a)^n}$$

$$\frac{M\,n!}{(s_0 - b)^n} \geq \frac{2^n\ M\,n!}{(b-a)^n} \tag{3.4}$$

Applying (3.3) and (3.4), we obtain

$$max\,|x(s)| \leq \frac{(b-a)^n}{2^n n!}\,max\,\left|x^{(n)}(s)\right|.$$

Thus we obtain

$$max\,|x(s)| \leq \frac{(b-a)^n}{2^n n!}\left\{x^{(n)}(s) + a_n(s)x(s) - a_n(s)x(s)\right\}$$

$$max\,|x(s)| \leq \frac{(b-a)^n}{2^n n!}\{\,max\,\left|x^{(n)}(s) + a_n(s)x(s)\right| + max\,|a_n(s)|\,max\,|x(s)|\,\}.$$

Now, let

$$\vartheta = \frac{(b-a)^n}{2^n n!}max\,|a_n(s)|\,,$$

Hence we get

$$max\,|x(s)| \leq \frac{(b-a)^n}{2^n n!\ (1-\vartheta)}\ \epsilon$$

choose K as $\frac{(b-a)^n}{2^n n!\,(1-\vartheta)}$, hence we obtain

$$max\,|x(s)| \leq K\,\epsilon.$$

Obviously, we obtain$y_0(s) = 0$ is a solution of$x^{(n)}(s) + a_n(s)x(s) = 0$ with boundary conditions $x(a) = x(b) = 0$, then $|x(s) - y_0(s)| \leq K\epsilon$. Hence (1.1) has the HUs stability property with boundary conditions (1.2).

Theorem 3.2.

If $max\,|a_n(s)| < \frac{n!}{(b-a)^n}$. Then equation (1.1) has the HUs with initial conditions (1.3).

Proof.

For every $\epsilon > 0$, $x \in C\,n\,[a,b]$, if $x^{(n)}(s) + a_n(s)x(s) \leq \epsilon$ and $x(a) = x'(a) = x''(a) = \ldots = x^{(n-1)}(a) = 0$.

Then by Taylor's formula, we obtain

$$x(s) = x(a) + x'(a)(s-a) + \frac{x''(a)}{2!}(s-a)^2 + \ldots + \frac{x^{(n)}(a)}{n!}(s-a)^n \tag{3.5}$$

Since we obtain$x(a) = x'(a) = x''(a) = \ldots = x^{(n-1)}(a) = 0$, then (3.5) becomes

$$x(s) = \frac{x^{(n)}(a)}{n!}(s-a)^n$$

take modulus on both sides, we obtain

$$|x(s)| = \left|\frac{x^{(n)}(a)}{n!}(s-a)^n\right|$$

$$max\,|x(s)| \leq max\ \left|x^{(n)}(s)\right|\frac{(b-a)^n}{n!}$$

so, we obtain

$$max\,|x(s)| \leq \frac{(b-a)^n}{n!}\ \{max\ \left|x^{(n)}(s) + a_n(s)x(s) - a_n(s)x(s)\right|\}$$

$$max\,|x(s)| \leq \frac{(b-a)^n}{n!}\ \{max\ \left|x^{(n)}(s) + a_n(t)x(t)\right| + max\,|a_n(s)|\,max\,|x(s)|\,\}.$$

$$max\,|x(s)| \leq \frac{(b-a)^n}{n!\ (1-\vartheta)}\ \epsilon$$

choose K as $\frac{(b-a)^n}{n!\,(1-\vartheta)}$, hence

$$max\,|x(s)| \leq K\,\epsilon.$$

Obviously, we obtain$y_0(s) = 0$ is a solution of $x^{(n)}(s) - a_n(s)x(s) = 0$ with initial conditions $y(a) = y'(a) = y''(a) = \ldots = y^{(n-1)}(a) = 0$, then

$$|x(s) - y_0(s)| \leq K\epsilon.$$

Hence (1.1) has the HUs stability property with initial conditions (1.3).

Corollary 1.

For every $\epsilon > 0$ and $x \in C^n(I)$ satisfies the inequality $|x''(s)| < |x'(s)| < |x(s)|$ if

$$max\ |a_n(s)| < \frac{n!}{(b-a)^n}$$

there exists a function $\phi: I \rightarrow [0, \infty)$ such that

$$\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon\,\phi(s),$$

with $x(a) = x'(a) = x''(a) = 0$. Then the linear differential equation (1) has the HyersUlam-Rassias stability with the initial conditions (3).

Corollary 2.

For every $\epsilon > 0$ and $x \in C^n(I)$ satisfies the inequality $|x''(s)| < |x'(s)| < |x(s)|$ if

$$max\ |a^n(s)| < \frac{n!\ 2^n}{(b-a)^n}$$

there exists a function $\phi: I \rightarrow [0, \infty)$ such that

$$\left|x^{(n)}(s) + a_n(s)x(s)\right| \leq \epsilon\,\phi(s),$$

with $x(a) = x(b) = 0$. Then the linear differential equation (1) has the HURs with boundary conditions (2).

III. Conclusion

We established one of the UHs, namely the HUs and HURs and of a n th order linear differential equation with initial and boundary conditions. Since this result corresponds to a general n th order differential equation, it will be more useful to readers to apply for various problems. Note that the HUs and HURs of a n th order nonlinear differential requires some more stronger conditions to prove the above Theorems. One can try similar type of result for nonlinear differential equations.

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